MA251 Algebra 1 - Week 2

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1 Week 2

Question 1.

Determine whether the following statements are true or false. If true, give a brief proof/justification. If false, provide a counterexample and brief justification for why it is a counterexample.

(a) If $n \times n$ matrices A, B satisfy $AB = 0_{n\times n}$, then at least one of A or B is a zero matrix.

(b) If A is an $n \times n$ matrix, then $\det(-A) = -\det(A)$.

(c) Let e_1, e_2, e_3, e_4 be the standard basis of $\mathbb{R}^4$. The vevtors $v_1 = e_1 - e_2$, $v_2 = e_2 - e_3$, $v_3 = e_3 - e_4$ form a basis for the subspace

$$V = \left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} \in \mathbb{R}^4 | x_1 + x_2 + x_3 + x_4 = 0 \right\}.$$

(d) The anticlockwise rotation of $\mathbb{R}^2$ by $\frac{\pi}{3}$ has no real eigenvalues.

Proof.

(a) It is false. Consider $n = 2$ and $A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, B $= \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$. Then

$$AB = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix},$$

but neither A nor B is a zero matrix.

(b) It is false. Consider $n = 2$ again, with $A = \begin{pmatrix} 1 & 0 \\ 0 & 1. \end{pmatrix}$

Hence, $\det(-A) = \det\left(\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}\right) = 1$. However $-\det\left(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right) = -1$, hence

$$\det(-A) \neq -\det(A).$$

(c) It is true. Compute v_1, v_2, v_3, we get

$$v_1 = \begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \end{pmatrix}, \quad v_2 = \begin{pmatrix} 0 \\ 1 \\ -1 \\ 0 \end{pmatrix}, \quad v_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ -1 \end{pmatrix}.$$

Hence we first check they are all in V.

Then we check the linear independence. Given $\alpha_1, \alpha_2, \alpha_3 \in \mathbb{R}$, and

$$\alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 v_3 = 0,$$

we get

$$\alpha_1 = \alpha_2 = \alpha_3 = 0.$$

Also, clearly 3 vectors cannot span $\mathbb{R}^4$, hence we need to show V has dimension of less than 4. We know that V has dimension at most 4 since it is a subspace of $\mathbb{R}^4$. By definition of V, we found that $e_1 \notin V$. Therefore V is a proper subspace of $\mathbb{R}^4$ and thus has dimension at most 3. Also v_1, v_2, v_3 span $\mathbb{R}^3$ and they form a basis of V.

(d) It is true. Recall the anticlockwise rotation matrix of $\mathbb{R}^2$ is defined by

$$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}.$$

Substitute $\theta = \frac{\pi}{3}$, we get

$$\begin{pmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix}.$$

To calculate the eigenvalues of this matrix, we follow the process:

$$\begin{aligned} \det\left(\begin{pmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix} - x\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right) &= 0 \\ \det\left(\begin{pmatrix} \frac{1}{2} - x & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} - x \end{pmatrix}\right) &= 0 \\ \left(\frac{1}{2} - x\right)^2 + \frac{3}{4} &= 0 \\ x^2 - x + 1 &= 0. \end{aligned}$$

For this quadratic function, $\Delta = 1 - 4 = -3 < 0$, so there are no real solutions to this function. Hence, the anticlockwise rotation of $\mathbb{R}^2$ by $\frac{\pi}{3}$ has no real eigenvalues.

□

Question 2.

Let V be the vector space of all polynomials of degree at most 3 with coefficients in $\mathbb{R}$, recall that we denote this by $\mathbb{R}[x]_{\leq 3}$ and let W be $\mathbb{R}[x]_{\leq 2}$. For the following linear maps $T : V \to W$, write the corresponding matrix A with respect to the ordered bases B_V and B_W in V and W.

(a) $T(p(x)) = p'(x+2)$ for $p \in V$, with $B_v = (1, x, x^2, x^3)$ and $B_W = (1, x, x^2)$.

(b) $T(p(x)) = (x-1)p''(2x)$ for $p \in V$, with $B_V = (1, x+1, x^2 - x, x^3 + x^2 + x)$ and $B_W = (1, x, x^2)$.

Solution.

(a) First we consider all the mappings of the elements in B_V.

$$\begin{aligned} T(1) &= 0 \\ T(x) &= 1 \\ T(x^2) &= 2(x+2) = 2x+4 \\ T(x^3) &= 3(x+2)^2 = 3x^2+12x+12. \end{aligned}$$

Since the mapping matrix is written to the left of the input vector, therefore in this case, we require a matrix that can be multiplied to the left of a 4×1 vector. Hence the matrix should be 3×4 instead of 4×3. Therefore, the corresponding matrix is

$$\begin{pmatrix} 0 & 1 & 4 & 12 \\ 0 & 0 & 2 & 12 \\ 0 & 0 & 0 & 3 \end{pmatrix}.$$

(b) Similarly, we get

$$\begin{aligned} T(1) &= 0 \\ T(x+1) &= 0 \\ T(x^2-x) &= 2(x-1) = 2x-2 \\ T(x^3+x^2+x) &= (x-1)(12x+2) = 12x^2-10x-2 \end{aligned}.$$

The matrix should be 3×4 with the same reason. Therefore the corresponding matrix is

$$\begin{pmatrix} 0 & 0 & -2 & -2 \\ 0 & 0 & 2 & -10 \\ 0 & 0 & 0 & 12 \end{pmatrix}.$$

□

Question 3.

Let $V = \mathbb{R}^2, W = \mathbb{R}^2$. We define ordered bases $\mathbf{E} = (\mathbf{e}_1, \mathbf{e}_2), \mathbf{E}' = (\mathbf{e}_1', \mathbf{e}_2')$ of V, and $\mathbf{F} = (\mathbf{f}_1, \mathbf{f}_2)$, $\mathbf{F}' = (\mathbf{f}_1', \mathbf{f}_2')$ of W by

$$\mathbf{e}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \mathbf{e}_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \quad \mathbf{e}_1' = \begin{pmatrix} 2 \\ 0 \end{pmatrix}, \mathbf{e}_2' = \begin{pmatrix} -1 \\ 3 \end{pmatrix},$$
$$\mathbf{f}_1 = \begin{pmatrix} -2 \\ -2 \end{pmatrix}, \mathbf{f}_2 = \begin{pmatrix} 2 \\ -1 \end{pmatrix}, \quad \mathbf{f}_1' = \begin{pmatrix} 0 \\ -1 \end{pmatrix}, \mathbf{f}_2' = \begin{pmatrix} 2 \\ 1 \end{pmatrix}.$$

It is known that the matrix of a linear map $T : V \to W$ with respect to the basis $\mathbf{E}$ in V and the basis $\mathbf{F}$ in W is given by

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}.$$

Compute the matrix of T with respect to the basis $\mathbf{E}'$ in V and the basis of $\mathbf{F}'$ in W.

Solution.

Recall that we have

$$M(T)^{\mathbf{F}'}_{\mathbf{E}'} = M(id_W)^{\mathbf{F}'}_{\mathbf{F}} \cdot M(T)^{\mathbf{F}}_{\mathbf{E}} \cdot M(id_V)^{\mathbf{E}}_{\mathbf{E}'}.$$

In our case, we know

$$M(T)^{\mathbf{F}}_{\mathbf{E}} = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}.$$

Solving simultaneously of the equations, we get the remaining two matrices

$$\begin{aligned} \mathbf{e}'_1 &= \begin{pmatrix} 2 \\ 0 \end{pmatrix} = a_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} + b_1 \begin{pmatrix} 1 \\ -1 \end{pmatrix} \\ \mathbf{e}'_2 &= \begin{pmatrix} -1 \\ 3 \end{pmatrix} = c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} + d_1 \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \end{aligned}$$

and

$$\begin{aligned} \mathbf{f}_1 &= \begin{pmatrix} -2 \\ -2 \end{pmatrix} = a_2 \begin{pmatrix} 0 \\ -1 \end{pmatrix} + b_2 \begin{pmatrix} 2 \\ 1 \end{pmatrix} \\ \mathbf{f}_2 &= \begin{pmatrix} 2 \\ -1 \end{pmatrix} = c_2 \begin{pmatrix} 0 \\ -1 \end{pmatrix} + d_2 \begin{pmatrix} 2 \\ 1 \end{pmatrix}. \end{aligned}$$

We obtained

$$\begin{cases} a_1 = 1 \\ b_1 = 1 \\ c_1 = 1 \\ d_1 = -2 \\ a_2 = 1 \\ b_2 = -1 \\ c_2 = 2 \\ d_2 = 1 \end{cases}.$$

Therefore,

$$M(id_V)^{\mathbf{E}}_{\mathbf{E}'} = \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & -2 \end{pmatrix}$$

and

$$M(id_W)^{\mathbf{F}'}_{\mathbf{F}} = \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ -1 & 1 \end{pmatrix}.$$

Therefore, the matrix of T with respect to the basis $\mathbf{E}'$ in V and the basis of $\mathbf{F}'$ in W is

$$M(T)^{\mathbf{F}'}_{\mathbf{E}'} = \begin{pmatrix} 1 & 2 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & -2 \end{pmatrix} = \begin{pmatrix} 17 & -13 \\ 4 & -2 \end{pmatrix}.$$

□

Question 4.

Find the eigenvalues and all corresponding eigenvectors of the following matrices.

(a)

$$A = \begin{pmatrix} 5 & 7 \\ 1 & -3 \end{pmatrix}.$$

(b)

$$B = \begin{pmatrix} 3 & 1 & -1 \\ -4 & -10 & 9 \\ -4 & -13 & 12 \end{pmatrix}.$$

Solution.

(a) We proceed using the following process:

$$\begin{aligned}\det(A - xI_2) = \det\left(\begin{pmatrix}5-x & 7\\ 1 & -3-x\end{pmatrix}\right) &= 0\\ x^2 - 2x - 22 &= 0\\ x_{1,2} &= 1 \pm \sqrt{23}.\end{aligned}$$

For the eigenvector, we following the format:

$$\begin{aligned}x_1\mathbf{v}_1 &= A\mathbf{v}_1\\ (1+\sqrt{23})\begin{pmatrix}a\\ b\end{pmatrix} &= \begin{pmatrix}5 & 7\\ 1 & -3\end{pmatrix}\begin{pmatrix}a\\ b\end{pmatrix}.\end{aligned}$$

Therefore we choose $b = 1$ and we get

$$\mathbf{v}_1 = \begin{pmatrix}4+\sqrt{23}\\ 1\end{pmatrix}$$

as the eigenvector of $x = 1 + \sqrt{23}$.

Similarly, for $x_2 = 1 - \sqrt{23}$, we get

$$\begin{aligned}x_2\mathbf{v}_2 &= A\mathbf{v}_2\\ (1-\sqrt{23})\begin{pmatrix}a\\ b\end{pmatrix} &= \begin{pmatrix}5 & 7\\ 1 & -3\end{pmatrix}\begin{pmatrix}a\\ b\end{pmatrix}\end{aligned}$$

This results in

$$\mathbf{v}_2 = \begin{pmatrix}4-\sqrt{23}\\ 1\end{pmatrix}$$

as the eigenvector of $x = 1 - \sqrt{23}$.

(b) Similarly we have

$$\begin{aligned}\det(B - xI_3) = \det\left(\begin{pmatrix}3-x & 1 & -1\\ -4 & -10-x & 9\\ -4 & -13 & 12-x\end{pmatrix}\right) &= 0\\ (3-x)(x^2-2x-3) - 4(1+x) - 4(-1-x) &= 0\\ (x-3)^2(x+1) &= 0\\ x_1 = x_2 &= 3\\ x_3 &= -1.\end{aligned}$$

Follow the same step,

$$\begin{aligned}x_1\mathbf{v}_1 &= B\mathbf{v}_1\\ 3\begin{pmatrix}a\\ b\\ c\end{pmatrix} &= \begin{pmatrix}3 & 1 & -1\\ -4 & -10 & 9\\ -4 & -13 & 12\end{pmatrix}\begin{pmatrix}a\\ b\\ c\end{pmatrix}\end{aligned}$$

and this results in

$$\begin{cases}a = -b\\ b = c.\end{cases}$$

Choose $b = 1$, we get

$$\mathbf{v}_1 = \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}$$

as the eigenvectors of eigenvalue 3.

For eigenvalue -1, we have

$$x_3\mathbf{v}_3 = B\mathbf{v}_3$$
$$-\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 3 & 1 & -1 \\ -4 & -10 & 9 \\ -4 & -13 & 12 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

and this results in

$$b = c.$$

Choose $b = c = 1$ and $a = 0$, we get

$$\mathbf{v}_3 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$$

as the eigenvector of eigenvalue 1.

□

Question 5.

For an $n \times n$ matrix we denote by A^t its transpose. Recall that $\det(A^t) = \det(A)$.

(a) Assume that A has real entries, that $n = 2m + 1$ is odd, and that $AA^t = I_n$, where I_n is the nn identity matrix. (A matrix satisfying the latter condition is called orthogonal). Show that A has an eigenvalue equal to $+1$ or -1.

(b) Suppose that $n = 2m + 1$ is odd, and that A is antisymmetric with real or complex entries, i.e. $A^t = -A$. Show $\det(A) = 0$.

Proof.

(a) Recall the relationship between eigenvalues and eigenvectors.

$$\lambda\mathbf{v}_1 = A\mathbf{v}_1 \implies (\lambda\mathbf{v}_1)^t = (A\mathbf{v}_1)^t.$$

Since if A is a square matrix, its eigenvalues are equal to the eigenvalues of its transpose, therefore

$$\lambda\mathbf{v}_1 = A\mathbf{v}_1$$
$$\lambda\mathbf{v}_2 = A^t\mathbf{v}_2.$$

Hence,

$$\lambda^2\mathbf{v}_1^t\mathbf{v}_1 = (A\mathbf{v}_1)^t(\lambda\mathbf{v}_1) = \mathbf{v}_1^t A^t A\mathbf{v}_1.$$

Since A is orthogonal, then $A^tA = AA^t = I_n$, hence

$$\lambda^2\mathbf{v}_1^t\mathbf{v}_1 = \mathbf{v}_1^t I_n\mathbf{v}_1 = \mathbf{v}_1^t\mathbf{v}_1.$$

Since $\mathbf{v}_1^t\mathbf{v}_1 \neq 0$. then $\lambda^2 = 1$ and therefore A has an eigenvalue equal to $+1$ or -1.

(b) Since $A^t = -A$, we have

$$\det(A) = \det(A^t) = \det(-A),$$

also recall that

$$\det(\alpha A) = \alpha^n \det(A)$$

where $n = \dim(A)$.

Therefore,

$$\det(A) = \det(-A) = (-1)^n \det(A) = -\det(A)$$

since $n = 2m + 1$ is odd. Therefore, we get

$$\det(A) = 0.$$

□