MA251 Algebra 1 - Week 8

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1 Week 8

Question 1.

Let V be a Euclidean vector space, and let $T : V \to V$ be a map that preserves distances:

$$|T(\mathbf{v}) - T(\mathbf{w})| = |\mathbf{v} - \mathbf{w}|, \quad \forall \mathbf{v}, \mathbf{w} \in V$$

and fixes $0 = 0_V \in V : T(0) = 0$. Here $|\mathbf{u}| = \sqrt{\mathbf{u} \cdot \mathbf{u}}$ as in the notes. Show that T is linear and orthogonal, in the following steps:

(a) Show that T preserves the norm : $|T(\mathbf{v})| = |\mathbf{v}|$ for all $\mathbf{v} \in V$.

(b) By expanding $|T(\mathbf{v}) - T(\mathbf{w})|^2$ show that T preserves the scalar product:

$$T(\mathbf{v}) \cdot T(\mathbf{w}) = \mathbf{v} \cdot \mathbf{w}$$

for all $\mathbf{v}, \mathbf{w} \in V$.

(c) Finally, considering

$$(T(\lambda\mathbf{v} + \mu\mathbf{w}) - \lambda T(\mathbf{v}) - \mu T(\mathbf{w})) \cdot (T(\lambda\mathbf{v} + \mu\mathbf{w}) - \lambda T(\mathbf{v}) - \mu T(\mathbf{w}))$$

prove that T is linear, hence orthogonal.

Proof.

(a) Let $\mathbf{w} = \mathbf{0}$, then by definition of T we have

$$|T(\mathbf{v}) - T(\mathbf{0})| = |\mathbf{v} - \mathbf{0}| = |\mathbf{v}|,$$

and since $T(0) = 0$, we have

$$|T(\mathbf{v}) - T(\mathbf{0})| = |T(\mathbf{v})| = |\mathbf{v}|.$$

(b) Expanding $|T(\mathbf{v}) - T(\mathbf{w})|^2$, we get

$$|T(\mathbf{v}) - T(\mathbf{w})|^2 = |T(\mathbf{v})|^2 - 2|T(\mathbf{v})| \cdot |T(\mathbf{w})| + |T(\mathbf{w})|^2.$$

Also we know by definition of T, we have

$$|\mathbf{v} - \mathbf{w}|^2 = |T(\mathbf{v}) - T(\mathbf{w})|^2.$$

Expanding LHS and using $T(\mathbf{v}) = |\mathbf{v}|$, we have

$$2\mathbf{v} \cdot \mathbf{w} = 2T(\mathbf{v}) \cdot T(\mathbf{w}) \implies T(\mathbf{v}) \cdot T(\mathbf{w}) = \mathbf{v} \cdot \mathbf{w}.$$

(c) Expand the expression and use (b), we get

$$\begin{aligned} I &= (T(\lambda\mathbf{v}+\mu\mathbf{w}) - \lambda T(\mathbf{v}) - \mu T(\mathbf{w})) \cdot (T(\lambda\mathbf{v}+\mu\mathbf{w}) - \lambda T(\mathbf{v}) - \mu T(\mathbf{w})) \\ &= |(\lambda\mathbf{v}+\mu\mathbf{w})|^2 - 2\lambda\mathbf{v}\cdot(\lambda\mathbf{v}+\mu\mathbf{w}) - 2\mu\mathbf{w}\cdot(\lambda\mathbf{v}+\mu\mathbf{w}) + \lambda^2|\mathbf{v}|^2 + 2\lambda\mu\mathbf{v}\cdot\mathbf{w} + \mu^2|\mathbf{w}|^2 \\ &= 0. \end{aligned}$$

Then $T((\lambda\mathbf{v}+\mu\mathbf{w}) - \lambda T(\mathbf{v}) - \mu T(\mathbf{w})) = 0$, and shows the linearity. Hence, it is orthogonal.

□

Question 2.

For the following symmetric matrices, find an orthonormal basis in which the matrix is diagonal.

$$A = \begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 1 & 3 \\ 1 & 3 & 1 \\ 3 & 1 & 1 \end{pmatrix}, \quad C = \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 2 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix}.$$

Solution.

(a) For A, the characteristic polynomial is $c_A(x) = (x-2)(x-4)$.

For $\lambda = 2$, we get the eigenvector $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$. For $\lambda = 4$, we get the eigenvector $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$. We need the length of the vectors to be 1 as well, so the basis is

$$\left\{ \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}, \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\}.$$

Check

$$\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix} \cdot \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = 0.$$

(b) Similarly, we find the characteristic polynomial to be $c_B(x) = (x-2)(x-5)(x+2)$.

For $\lambda = 2$, the eigenvector is $\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$. For $\lambda = 5$, eigenvector is $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$. For $\lambda = -2$, the eigenvector is $\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$.

Similarly, every vector should have length 1, so the basis is

$$\left\{ \frac{1}{\sqrt{6}}\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}, \frac{1}{\sqrt{3}}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \frac{1}{\sqrt{2}}\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \right\}.$$

Check

$$\frac{1}{\sqrt{6}}\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} \cdot \frac{1}{\sqrt{3}}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \cdot \frac{1}{\sqrt{2}}\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} = 0.$$

(c) The characteristic polynomial is $c_C(x) = x(x-2)^3$.

For $\lambda = 0$, the eigenvector is $\begin{pmatrix} 0 \\ 1 \\ 0 \\ -1 \end{pmatrix}$. For $\lambda = 2$, the eigenvector is $\begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix}$.

Similarly, every vector should have length 1, so the basis can be

$$\left\{ \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ 0 \\ -1 \end{pmatrix}, \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix} \right\}.$$

Check

$$\frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ 0 \\ -1 \end{pmatrix} \cdot \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix} = 0.$$

□

Question 3.

Classify the following curves and surfaces (say whether it is an ellipse, a parabola, a two-sheeted hyperboloid etc.):

(a) $x^2 - y^2 + 2xy - 1 = 0$(2 dimensions);

(b) $x^2 - y^2 + 2xy - 1 = 0$(3 dimensions);

(c) $x^2 + 2xy + y^2 + x + 1 = 0$(2 dimensions);

(d) $x^2 + y^2 - 2z^2 - x - y - 4z = 0$(3 dimensions);

(e) $x^2 + y^2 - z^2 + 6x - 4y - 2z + 12 = 0$(3 dimensions);

(f) $x^2 + y^2 - z^2 + 2xy - 2xz - 2yz - y = 0$(3 dimensions)

Solution.

(a) We rewrite it as

$$\begin{aligned} x^2 - y^2 + 2xy - 1 = (x+y)^2 - 2y^2 - 1 = 0 \\ (x+y)^2 - 2y^2 = 1. \end{aligned}$$

It is a hyperbola.

(b) We rewrite it as the same form in (a), and it is a hyperbolic cylinder.

(c) We rewrite it as

$$x^2 + 2xy + y^2 + x + 1 = (x+y)^2 + x + 1 = 0.$$

It is a hyperbola.

(d) We rewrite it as

$$\begin{aligned} x^2 + y^2 - 2z^2 - x - y - 4z = 0 \\ -(x - \frac{1}{2})^2 - (y - \frac{1}{2})^2 + 2(z+1)^2 = \frac{3}{2}. \end{aligned}$$

We look in the notes and see we must have a hyperboloid of two sheets.

(e) We rewrite it as

$$\begin{aligned} x^2 + y^2 - z^2 + 6x - 4y - 2z + 12 &= 0 \\ (x+3)^2 + (y-2)^2 - (z+1)^2 &= 0. \end{aligned}$$

It is an elliptical cone.

(f) We rewrite it as

$$\begin{aligned} x^2 + y^2 - z^2 + 2xy - 2xz - 2yz - y &= 0 \\ (x+y-z)^2 - 2z^2 - y &= 0. \end{aligned}$$

It is a hyperbolic paraboloid.

□