MA251 Algebra 1 - Week 7

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April 1, 2023

1 Week 7

Question 1.

Let q denote the quadratic form on $\mathbb{R}^3$ defined by

$$q(x, y, z) = 8xy - yz + 3y^2 - z^2.$$

Find a basis of $\mathbb{R}^3$ in which the matrix of q is diagonal. What is the signature of q?

Solution.

The matrix in the standard basis is

$$A = \begin{pmatrix} 0 & 4 & 0 \\ 4 & 3 & -\frac{1}{2} \\ 0 & -\frac{1}{2} & -1 \end{pmatrix}.$$

Then we follow the algorithms. We see that $a_{11} = 0$ while a_{22} and $a_{33} \neq 0$, so we interchange $\mathbf{b}_1$ and $\mathbf{b}_2$, and the new basis are

$$\mathbf{b}_1 = \mathbf{e}_2, \quad \mathbf{b}_2 = \mathbf{e}_1, \quad \mathbf{b}_3 = \mathbf{e}_3.$$

The change of basis matrix is

$$P = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

The new matrix is therefore

$$\begin{aligned} B &= P^T A P \\ &= \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 4 & 0 \\ 4 & 3 & -\frac{1}{2} \\ 0 & -\frac{1}{2} & -1 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 3 & 4 & -\frac{1}{2} \\ 4 & 0 & 0 \\ -\frac{1}{2} & 0 & -1 \end{pmatrix}. \end{aligned}$$

Now $a_{11} \neq 0$, and we want $\tau(\mathbf{b}_1, \mathbf{b}_i) = 0$, for all $i > 1$. Hence replace $\mathbf{b}_i$ with $\mathbf{b}_i - \frac{a_{1i}}{a_{11}}\mathbf{b}_1$, and the new basis are

$$\mathbf{b}'_1 = \mathbf{b}_1 = \mathbf{e}_2, \quad \mathbf{b}'_2 = \mathbf{b}_2 - \frac{a_{12}}{a_{11}}\mathbf{b}1 = \mathbf{e}_1 - \frac{4}{3}\mathbf{e}_2, \quad \mathbf{b}'_3 = \mathbf{b}_3 - \frac{a_{13}}{a_{11}}\mathbf{b}_1 = \mathbf{e}_3 + \frac{1}{6}\mathbf{e}_2.$$

Now the change of basis is

$$Q = \begin{pmatrix} 1 & -\frac{a_{12}}{a_{11}} & -\frac{a_{13}}{a_{11}} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & -\frac{4}{3} & \frac{1}{6} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Combining the two change of basis matrices:

$$\begin{aligned} P' &= PQ \\ &= \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & -\frac{4}{3} & \frac{1}{6} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 0 & 1 & 0 \\ 1 & -\frac{4}{3} & \frac{1}{6} \\ 0 & 0 & 1 \end{pmatrix}. \end{aligned}$$

Hence the matrix with respect to the new basis is

$$\begin{aligned} B' &= (P')^T A P' \\ &= \begin{pmatrix} 0 & 1 & 0 \\ 1 & -\frac{4}{3} & 0 \\ 0 & \frac{1}{6} & 1 \end{pmatrix} \begin{pmatrix} 0 & 4 & 0 \\ 4 & 3 & -\frac{1}{2} \\ 0 & -\frac{1}{2} & -1 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ 1 & -\frac{4}{3} & \frac{1}{6} \\ 0 & 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 3 & 0 & 0 \\ 0 & -\frac{16}{3} & \frac{2}{3} \\ 0 & \frac{2}{3} & -\frac{13}{12} \end{pmatrix}. \end{aligned}$$

Now we focus on the smaller block. Check that the down right corner block's $a_{11} \neq 0$ or not. Note that $a_{11} \neq 0$, and hence we are in step 2. Replace b_i with $b_i - \frac{a_{2i}}{a_{22}}\mathbf{b}_2$, and the new basis are

$$\mathbf{b}_1'' = \mathbf{b}_1 = \mathbf{e}_2, \quad \mathbf{b}_2'' = \mathbf{b}_2' = \mathbf{e}_1 - \frac{4}{3}\mathbf{e}_2, \quad \mathbf{b}_3'' = \mathbf{b}_3' - \frac{a_{23}}{a_{22}}\mathbf{b}_2' = \mathbf{e}_3 + \frac{1}{6}\mathbf{e}_2 + \frac{1}{8}(\mathbf{e}_1 - \frac{4}{3}\mathbf{e}_2) = \frac{1}{8}\mathbf{e}_1 + \mathbf{e}_3.$$

The change of basis is

$$\begin{aligned} Q' &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -\frac{a_{23}}{a_{22}} \\ 0 & 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & \frac{1}{8} \\ 0 & 0 & 1 \end{pmatrix}. \end{aligned}$$

Combining the change of basis matrices, we get

$$\begin{aligned} P'' &= P'Q' \\ &= \begin{pmatrix} 0 & 1 & 0 \\ 1 & -\frac{4}{3} & \frac{1}{6} \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & \frac{1}{8} \\ 0 & 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 0 & 1 & \frac{1}{8} \\ 1 & -\frac{4}{3} & 0 \\ 0 & 0 & 1 \end{pmatrix}. \end{aligned}$$

Hence, the new matrix in the new basis is

$$\begin{aligned} C &= (P'')^T A P'' \\ &= \begin{pmatrix} 0 & 1 & 0 \\ 1 & -\frac{4}{3} & 0 \\ \frac{1}{8} & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 4 & 0 \\ 4 & 3 & -\frac{1}{2} \\ 0 & -\frac{1}{2} & -1 \end{pmatrix} \begin{pmatrix} 0 & 1 & \frac{1}{8} \\ 1 & -\frac{4}{3} & 0 \\ 0 & 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 3 & 0 & 0 \\ 0 & -\frac{16}{3} & 0 \\ 0 & 0 & -1 \end{pmatrix}. \end{aligned}$$

Now the matrix is diagonal and hence the basis is

$$\mathcal{B}'' = \left\{ \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ -\frac{4}{3} \\ 0 \end{pmatrix}, \begin{pmatrix} \frac{1}{8} \\ 0 \\ 1 \end{pmatrix} \right\}.$$

Since the diagonal matrix C has 1 positive eigenvalue and 2 negative eigenvalues (eigenvalues of diagonal matrices are their diagonal entries), its signature is $(1, 2)$.

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Question 2.

Find a QR-decomposition of the following matrices

$$A = I_4, \quad B = 3I_4, \quad C = \begin{pmatrix} 1 & 2 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix}.$$

Solution.

(a) The first one is already in QR-decomposition as A is self-orthogonal.

(b) The second one is easy to spot as well.

$$B = QR = I_4 3I_4.$$

(c) Let $\mathbf{g}_1, \mathbf{g}_2, \mathbf{g}_3$ be three columns of C. Then we apply the Gram-Schmidt process:

We have $|\mathbf{g}_1| = \sqrt{2}$, so

$$\mathbf{f}_1 = \frac{\mathbf{g_1}}{|\mathbf{g}_1|} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}.$$

For next step, we take $\mathbf{f}_2' = \mathbf{g}_2 - (\mathbf{f}_1 \cdot \mathbf{g}_2)\mathbf{f}_1$.

$$\mathbf{f}_2' = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} - \frac{2}{\sqrt{2}} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$$

and so

$$\mathbf{f}_2 = \frac{\mathbf{f}_2'}{|\mathbf{f}_2'|} = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}.$$

For the final step, take $\mathbf{f}_3' = \mathbf{g}_3 - (\mathbf{f}_1 \cdot \mathbf{g}_3)\mathbf{f}_1 - (\mathbf{f}_2 \cdot \mathbf{g}_3)\mathbf{f}_2$.

$$\mathbf{f}_3' = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} - \frac{1}{\sqrt{2}} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} - 0 = \begin{pmatrix} -\frac{1}{2} \\ 1 \\ frac12 \end{pmatrix}$$

and so

$$\mathbf{f}_3 = \frac{\mathbf{f}_3'}{|\mathbf{f}_3'|} = \frac{1}{\sqrt{6}} \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix}.$$

Hence,

$$Q = \frac{1}{\sqrt{6}} \begin{pmatrix} \sqrt{3} & \sqrt{2} & -1 \\ 0 & \sqrt{2} & 2 \\ \sqrt{3} & -\sqrt{2} & 1 \end{pmatrix}.$$

To determine R, we can set

$$R = \begin{pmatrix} a & b & c \\ 0 & d & e \\ 0 & 0 & f \end{pmatrix}$$

and times Q to get C then compare coefficients. Therefore we have

$$R = \begin{pmatrix} \sqrt{2} & \sqrt{2} & \frac{1}{\sqrt{2}} \\ 0 & \sqrt{3} & 0 \\ 0 & 0 & \frac{\sqrt{6}}{2} \end{pmatrix}.$$

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