MA251 Algebra 1 - Week 6

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1 Week 6

Question 1.

Determine which of the following maps $\tau : V \times V \to \mathbb{R}$ are bilinear forms for $V = \mathbb{R}^3$. For those that are, determine whether they are symmetric or not, and find their left and right radicals.

(a) $\tau(\underline{x}, \underline{y}) = x_1y_1 + x_2y_2 + x_3y_3 + x_1y_3$,

(b) $\tau(\underline{x}, \underline{y}) = x_1y_1 + x_2^2y_2 + x_3y_3 + x_1y_3$,

(c) $\tau(\underline{x}, \underline{y}) = x_1y_1 + x_1y_3 + y_1x_3$,

(d) $\tau(\underline{x}, \underline{y}) = x_1y_2 + y_1x_2 + x_1y_3 + y_1x_3 + x_2y_2 + x_3y_3$.

Solution.

(a) It is a bilinear form. The matrix is

$$A = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

It is not symmetric as the matrix is not symmetric. Or equivalently, take $\underline{x} = \mathbf{e}_1$ and $\underline{y} = \mathbf{e}_3$, then we see that

$$\tau(\underline{x}, \underline{y}) = \tau(\mathbf{e}_1, \mathbf{e}_3) = 1 \neq \tau(\mathbf{e}_3, \mathbf{e}_1) = 0.$$

By definition, the left radical is the kernel of A^T and the right radical is the kernel of A. Since both A and A^T are full rank, hence both left and right radicals are $\mathbf{0}$.

(b) It is not a bilinear form. Note that choose $\underline{x} = \underline{y} = \mathbf{e}_2$, then

$$\tau(2\mathbf{e}_2, \mathbf{e}_2) = 4 \neq 2\tau(\mathbf{e}_2, \mathbf{e}_2) = 2.$$

(c) It is a bilinear form. The matrix is

$$A = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

It is symmetric since the matrix is symmetric. Thus, the left and right radicals are the same. Therefore,

$$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix},$$

indicating

$$\ker(A) = \ker(A^T) = \begin{pmatrix} 0 \\ b \\ 0 \end{pmatrix}.$$

Hence, the left and right radicals are the span of $\mathbf{e}_2$.

(d) It is a bilinear form. The matrix represented is

$$A = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}.$$

It is symmetric as the matrix is symmetric. Again, by finding the kernels,

$$\begin{pmatrix} 0 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix},$$

indicating

$$\ker(A) = \ker(A^T) = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

Hence, the left and right radicals are the zero subspace. (Alternatively since the rank of A is 3).

□

Question 2.

For the following bilinear forms write down their corresponding matrix with respect to the standard basis $\mathcal{B}$. Then write down the matrix with respect to the basis $\mathcal{B}'$. Are the forms symmetric? Are they non-degenerate?

(a) $V = \mathbb{R}^2$, with $\mathcal{B} = e_1, e_2, \mathcal{B}' = e_1 + 3e_2, e_2 - 2e_1$ and $\tau(\underline{x}, \underline{y}) = x_1y_1 + x_1y_2 + 2y_1x_2$,

(b) $V = \mathbb{R}[x]_{\leq 3}$, with $\mathcal{B} = 1, x, x^2, x^3$, $\mathcal{B}' = 1, x+1, x^2+1, x^3+1$ and $\tau(f, g) = 3a_0b_0 + 2a_1b_2 + a_1b_3 + a_3b_1 + 2a_2b_1$ for $f = \sum_i a_ix^i$ and $g = \sum_i b_ix^i$.

Solution.

(a) We see that the matrix representing τ is

$$A = \begin{pmatrix} 1 & 1 \\ 2 & 0 \end{pmatrix},$$

which is not symmetric as A is not. The rank of A is 2, hence A is non-degenerate.

The change of basis is

$$P = \begin{pmatrix} 1 & -2 \\ 3 & 1 \end{pmatrix}.$$

Hence the matrix with respect to the basis $\mathcal{B}'$ is

$$\begin{aligned} B &= P^TAP \\ &= \begin{pmatrix} 1 & 3 \\ -2 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 2 & 0 \end{pmatrix} \begin{pmatrix} 1 & -2 \\ 3 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 10 & -13 \\ -6 & -2 \end{pmatrix}. \end{aligned}$$

(b) The matrix representing τ is

$$A = \begin{pmatrix} 3 & 0 & 0 & 0 \\ 0 & 0 & 2 & 1 \\ 0 & 2 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix},$$

which is symmetric as $A = A^T$. The rank of A is 3, so A is not non-degenerate.

The change of basis is

$$P = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Hence the matrix with respect to the basis $\mathcal{B}'$ is

$$\begin{aligned} B &= P^T A P \\ &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 3 & 0 & 0 & 0 \\ 0 & 0 & 2 & 1 \\ 0 & 2 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 3 & 3 & 3 & 3 \\ 3 & 3 & 5 & 4 \\ 3 & 5 & 3 & 3 \\ 3 & 4 & 3 & 3 \end{pmatrix}. \end{aligned}$$

□

Question 3.

Find the signatures of the following real quadratic forms on $\mathbb{R}^2$, where $v = \begin{pmatrix} x \\ y \end{pmatrix}$.

$$q_1(v) = -x^2 + y^2, \quad q_2(v) = 4xy + y^2, \quad q_3(v) = xy.$$

Solution.

(a) The first matrix is

$$A = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix},$$

it is already diagonal, and we can read the signature is $(1, 1)$.

(b) The second matrix is

$$B = \begin{pmatrix} 0 & 2 \\ 2 & 1 \end{pmatrix},$$

the signature is determined by the number of positive and negative eigenvalues of the matrix and we find that the eigenvalues for B are

$$\begin{cases} x_1 = \frac{1+\sqrt{17}}{2} \\ x_2 = \frac{1-\sqrt{17}}{2} \end{cases},$$

so there is one positive and one negative eigenvalue and the signature is $(1, 1)$ again.

(c) The third matrix is

$$C = \begin{pmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix},$$

the signature is still be determined when we find out the eigenvalues of C:

$$\begin{cases} x_1 = \frac{1}{2} \\ x_2 = -\frac{1}{2} \end{cases},$$

so the signature is $(1, 1)$ again.

□

Question 4.

Show that the set of all bilinear forms on a vector space V (cal them $Bil(v)$) has a structure of a vector space itself by defining a natural addition and scalar multiplication on them. (Hint: if you are stuck, think about the structure we have on matrices and our map between bilinear forms on K^n and $n \times n$ matrices).

Show that the set of symmetric bilinear forms form a subspace of $Bil(V)$ of dimension $n + \frac{n(n-1)}{2}$.

Proof.

Let B and C be bilinear forms and $\alpha \in K$. We define $B + C(x, y) = B(x, y) + C(x, y)$ and $(\alpha B)(x, y) = \alpha B(x, y)$ for all $x, y \in V$.This corresponds with the addition and scalar multiplication of $n \times n$ matrices.

To check the dimension, looking at symmetric matrices. Vaguely, we have n choices from the diagonals, and then $\frac{n(n-1)}{2}$ for the entries above the diagonal because the ones below the diagonal must be the same. Hence the dimension is $n + \frac{n(n-1)}{2}$. □