MA244 Analysis III Support Class - Week 10

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1 Week 10

Question 1.

Let f be an analytic function in a region $\Omega \subseteq \mathbb{C}$ and let $a \in \Omega$. If the ball $B_r(a) \subset \Omega$, then show that

$$f(a) = \frac{1}{2\pi} \int_0^{2\pi} f(a + re^{i\theta}) \mathrm{d}\theta.$$

This result is called the mean value theorem.

Proof.

Using Cauchy's formula, since f is analytic in Ω where $B_r(a) \subset \Omega$ is a positively oriented simple closed C^1 curve, then

$$f(a) = \frac{1}{2\pi i} \int_{B_r(a)} \frac{f(z)}{z - a} \mathrm{d}w.$$

Parametrise $B_r(a)$, we have $z = a + re^{i\theta}$, where $\theta \in [0, 2\pi]$. This is a circle with radius r centered at a, and therefore $\frac{\mathrm{d}z}{\mathrm{d}\theta} = ire^{i\theta}$. We have

$$f(a) = \frac{1}{2\pi i} \int_0^{2\pi} \frac{f(a + re^{i\theta})}{a + re^{i\theta} - a} ire^{i\theta} \mathrm{d}\theta = \frac{1}{2\pi} \int_0^{2\pi} f(a + re^{i\theta}) \mathrm{d}\theta.$$

□

Question 2.

Let f be a non-constant analytic function in $B_r(z)$ for some $r > 0$ and $z \in \mathbb{C}$. Prove that there exists $z' \in B_r(z)$ such that

$$\left|f(z')\right| > |f(z)|.$$

This lemma is the main technical ingredient for the proof of the so called maximum modulus theorem, stating that the modulus of a non-constant analytic function cannot have a point of local maximum.

Hint: You may want to use the mean value theorem from Q1 for the proof.

Proof.

Use the result in (a), we have

$$f(z) = \frac{1}{2\pi} \int_0^{2\pi} f(z + r' e^{i\theta}) \mathrm{d}\theta.$$

Hence,

$$|f(z)| = \left|\frac{1}{2\pi}\int_0^{2\pi} f(z+r'e^{i\theta})\mathrm{d}\theta\right| \le \frac{1}{2\pi}\int_0^{2\pi}\left|f(z+r'e^{i\theta})\right|\mathrm{d}\theta \le \frac{1}{2\pi}\int_0^{2\pi}\max_\theta\left|f(z+r'e^{i\theta})\right|\mathrm{d}\theta.$$

Here, $\max_\theta \left|f(z+r'e^{i\theta})\right|$ is just a constant, so we can put it outside the integral:

$$|f(z)| \le \frac{1}{2\pi}\max_\theta\left|f(z+r'e^{i\theta})\right|\int_0^{2\pi}\mathrm{d}\theta = \max_{|z'-z|=r'}\left|f(z')\right|.$$

Now after we get this relation, we prove by contradiction. Suppose for all z, we have $|f(z')| \le |f(z)|$. Then we have

$$|f(z)| \le \max_{|z'-z|=r'}\left|f(z')\right| \le |f(z)|,$$

which implies $|f(z)| = \max_{|z'-z|=r'}\left|f(z')\right|$.

Since f is non-constant, then there exists z'' such that $|f(z'')| > |f(z)|$. Let $r'' = |z''-z|$. By continuity of f, there exists a point z' on the circle $|z'-z| = r'$ such that

$$|f(z')| = |f(z'')|,$$

which in turn $|f(z')| > |f(z)|$, and it contradicts with our assumption. Therefore, there exists $z' \in B_r(z)$ such that

$$\left|f(z')\right| > |f(z)|.$$

□

Question 3.

Let $f : \mathbb{C} \to \mathbb{C}$ be an entire function that satisfies $|f(z)| \le K|z|^k$ for all $z \in \mathbb{C}$ for some fixed $K > 0$ and some positive integer $k \ge 0$. Prove that f is a polynomial of degree at most k. (Note that $k = 0$ is Liouville's theorem to be discussed in Week 10.)

Proof.

By Taylor Series Expansion, since f is an entire function, it can be represented as

$$f(z) = \sum_{n=0}^{\infty} c_n z^n,$$

where the coefficients c_n are given by

$$c_n = \frac{1}{2\pi i}\int_\gamma \frac{f(w)}{w^{n+1}}\mathrm{d}w = \frac{f^{(n)}(0)}{n!}.$$

We are expanding at the origin, so

$$c_n = \frac{f^{(n)}(0)}{n!}.$$

As a result,

$$f^{(n)}(0) = \frac{n!}{2\pi i}\int_\gamma \frac{f(w)}{w^{n+1}}\mathrm{d}w.$$

Let $\gamma = \partial B_R(0)$ which oriented counter-clockwisely, for some $R > 0$ we have

$$\begin{aligned}\left|f^{(n)}(0)\right| &= \left|\frac{n!}{2\pi i}\int_0^{2\pi}\frac{f(Re^{it})}{(Re^{it})^{n+1}}Rie^{it}\mathrm{d}t\right| \\ &= \left|\frac{n!}{2\pi}\int_0^{2\pi}\frac{f(Re^{it})}{(Re^{it})^{n}}\mathrm{d}t\right| \\ &\leq \frac{n!}{2\pi}\int_0^{2\pi}\left|\frac{f(Re^{it})}{(Re^{it})^{n}}\mathrm{d}t\right| \\ &= \frac{n!}{2\pi}\int_0^{2\pi}\frac{\left|f(Re^{it})\right|}{R^n}\mathrm{d}t.\end{aligned}$$

Since $|f(z)| \leq K\,|z|^k$ for all $z \in \mathbb{C}$ for some fixed $K > 0$ and some positive integer $k \geq 0$, then

$$\left|f^{(n)}(0)\right| \leq \frac{n!}{2\pi}\int_0^{2\pi}\frac{\left|f(Re^{it})\right|}{R^n}\mathrm{d}t \leq \frac{n!}{2\pi}\int_0^{2\pi}\frac{K\left|Re^{it}\right|^k}{R^n}\mathrm{d}t = n!\frac{KR^k}{R^n}.$$

Since R is arbitrary, we can take the limit as R goes to infinity, therefore for $n > k$,

$$\left|f^{(n)}(0)\right| \leq 0 \implies f^{(n)}(0) = 0.$$

Therefore

$$f(z) = \sum_{n=0}^{k} c_n z^n,$$

meaning it is a polynomial of degree at most k. □