# MA244 Analysis III Support Class - Week 9

Louis Li

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## 1 Week 9

**Question 1.**

Integrate $f(z) = z^2$ over the closed curve $C$, consisting of two radial lines of unit length and a wedge of a unit circle, as shown in the adjoining figure. The arrow in the figure shows the orientation of $C$.

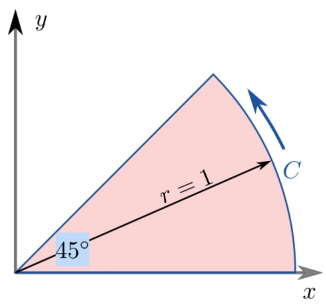


Figure 1: Question 1

*Solution.*

There are three parts of this curve, the two radial lines and a wedge of a unit circle. We parametrise each part:

(a) The first part is a radial that lies on the $x$-axis. This parametrisation is $z = t$, and the integral is

$$I_1 = \int_0^1 z^2 \mathrm{d}z = \int_0^1 t^2 \mathrm{d}t = \frac{1}{3}.$$

(b) The second part is a wedge of a unit circle with 45 degrees. This parametrisation is $z = e^{it}$, and $\frac{\mathrm{d}z}{\mathrm{d}t} = ie^{it}$. Therefore, the integral is

$$I_2 = \int_0^{\frac{\pi}{4}} z^2 \mathrm{d}z = \int_0^{\frac{\pi}{4}} \left(e^{it}\right)^2 \cdot ie^{it} \mathrm{d}t = \frac{e^{\frac{3\pi}{4}i} - 1}{3}.$$

(c) The third part is again a radial that makes 45 degrees with the $x$-axis. This parametrisation is $z = (1-t)e^{i\frac{\pi}{4}}$. The reason for this parametrisation is that as $t \in [0, 1]$, $|z|$ moves from 1 to 0 (mainly contributed by $(1-t)$), and since $z = x + iy$, we can rewrite it as $z = e^{i\theta}$. Here $\theta = \frac{\pi}{4}$, and hence $z = (1-t)e^{i\frac{\pi}{4}}$. We have $\frac{\mathrm{d}z}{\mathrm{d}t} = -e^{\frac{i\pi}{4}}$ and $z^2 = (1-t)^2 e^{i\frac{\pi}{2}}$. Therefore, the integral is

$$I_3 = \int_0^1 z^2 \mathrm{d}z = \int_0^1 -e^{\frac{i\pi}{4}}(1-t)^2 e^{i\frac{\pi}{2}} \mathrm{d}t = -\frac{e^{i\frac{3\pi}{4}}}{3}.$$

Therefore,

$$I = I_1 + I_2 + I_3 = 0.$$

Alternatively, we can use Cauchy's Theorem: Let $f : \Omega \to \mathbb{C}$ be an analytic function, with $\Omega$ an open, simply connected domain. Let $\gamma$ be a $C^1$ closed curve in $\Omega$. Then

$$\int_\gamma f(z)\mathrm{d}z = 0.$$

Since $f(z) = z^2$ is analytic and $\gamma$ here is a closed curve, we obtain the result using Cauchy's theorem. □

**Question 2.**

Let $f$ be a continuous function defined over $\mathbb{C}$ with the property that

$$\lim_{z\to\infty} zf(z) = k.$$

Let $D$ be the curve $z = Re^{it}$, with $0 \le t \le \theta$ oriented counter-clockwise.

(a) Show that

$$\lim_{R\to\infty} \int_D f(z)\mathrm{d}z = i\theta k$$

(b) Use this result to evaluate

$$\lim_{R\to\infty} \int_C \frac{\mathrm{d}z}{\sqrt{z^2-1}},$$

where $C$ is a circle of radius $R$ oriented counter-clockwise.

*Proof.*

(a) The property that

$$\lim_{z\to\infty} zf(z) = k$$

implies that for every $\varepsilon > 0$, there exists a $N > 0$ such that for every $|z| > N$,

$$|zf(z) - k| < \frac{\varepsilon}{\theta}.$$

With $z = Re^{it}$, we have $\frac{\mathrm{d}z}{\mathrm{d}t} = iRe^{it}$.

Therefore, for $R > N$,

$$\begin{aligned}\left|\int_D f(z)\mathrm{d}z - i\theta k\right| &= \left|\int_D f(z)\mathrm{d}z - k\int_0^\theta \frac{iRe^{it}}{Re^{it}}\mathrm{d}t\right| \\ &= \left|\int_D f(z)\mathrm{d}z - k\int_D \frac{\mathrm{d}z}{z}\right| \\ &= \left|\int_D \frac{zf(z)-k}{z}\mathrm{d}z\right| \\ &\leq \int_0^\theta \left|\frac{zf(z)-k}{z}\right| |\mathrm{d}z| \\ &= \int_0^\theta |zf(z)-k| \left|\frac{1}{z}\right| |iz\mathrm{d}t| \\ &= \int_0^\theta |zf(z)-k|\,\mathrm{d}t \\ &< \int_0^\theta \frac{\varepsilon}{\theta}\mathrm{d}t \\ &= \varepsilon.\end{aligned}$$

By definition of limits, this means

$$\lim_{R\to\infty}\int_D f(z)\mathrm{d}z = i\theta k.$$

(b) Here $f(z) = \frac{1}{\sqrt{z^2-1}}$. Hence,

$$\lim_{z\to\infty} zf(z) = \lim_{z\to\infty}\frac{z}{\sqrt{z^2-1}} = 1.$$

Therefore, by $(a)$, we have

$$\lim_{R\to\infty}\int_C \frac{\mathrm{d}z}{\sqrt{z^2-1}} = i2\pi = 2\pi i.$$

□

**Question 3.**

Assume that $f : \mathbb{C} \to \mathbb{C}$ is analytic. Prove that if $|f|$ is also analytic then $f$ must be a constant.

*Proof.*

Suppose $f = u + iv$. If $f$ is analytic then $|f|$ is also analytic and in turn, $|f|^2$ is also analytic, meaning that $g = u^2 + v^2$ is analytic. Using Cauchy-Riemann equations, we have

$$u_x = v_y, \qquad u_y = -v_x.$$

Therefore, in this case $u' = u^2 + v^2$ and $v' = 0$, and

$$2uu_x + 2vv_x = 0 \qquad -2uv_x + 2vu_x = 0,$$

which means $(u^2+v^2)_x = 0 = (u^2+v^2)_y$. Therefore $g = \left|f^2\right|$ is a constant and $|f| = \sqrt{g}$ is a constant. If $|f| = 0$, then $f = 0$ and we are done. If $|f| > 0$, let's denote this as $\rho$. Since $f$ is analytic, we also apply the Cauchy-Riemann equations to the above equation,

$$\begin{cases} uu_x - vu_y &= 0 \\ uu_y + vv_y &= 0. \end{cases}$$

By canceling $u_y$, we get

$$u^2u_x + v^2v_y = u^2u_x + v^2u_x = (u^2+v^2)u_x = 0.$$

By canceling $u_x$, we get

$$(u^2+v^2)v_x = 0.$$

Therefore, we get

$$(u^2+v^2)u_x = (u^2+v^2)v_x.$$

For the non-trivial solution, we get $u_x = v_x = 0$, and $u_x = v_x = v_y = v_x = 0$. Therefore, $f$ is a constant.

□

**Question 4.**

Let $f : z \to \sum_{n=0}^{\infty} a_n z^n$, where the series in the RHS has a radius of convergence $R > 0$. For any $r \in (0, R)$ calculate

$$I = \frac{1}{2\pi i}\int_{\partial B_r(0)} \frac{|f(z)|^2}{z}\mathrm{d}z,$$

the average value of $|f|^2$ over the circle of radius $r$ centered on the origin. The contour of integration is oriented counter-clockwise.

*Solution.*

Since the integral is about calculating the average value of $|f|^2$ over the circle of radius $r$ centered on the origin. Therefore, let's consider the parametrisation $z = re^{it}$, where $t \in [0, 2\pi]$ and $\frac{\mathrm{d}z}{\mathrm{d}t} = ire^{it}$.

Therefore,

$$\begin{aligned} I_k &= \frac{1}{2\pi i}\int_0^{2\pi} \frac{\left|\sum_{n=0}^{\infty} a_n \left(re^{it}\right)^n\right|^2}{re^{it}} \cdot ire^{it}\mathrm{d}t \\ &= \frac{1}{2\pi}\int_0^{2\pi} \left|\sum_{n=0}^{\infty} a_n r^n e^{int}\right|^2 \mathrm{d}t. \end{aligned}$$

Here we uses the modulus property where $|w|^2 = w\overline{w}$,

$$\begin{aligned}
I_k &= \frac{1}{2\pi}\int_0^{2\pi}\left|\sum_{n=0}^{\infty} a_n r^n e^{int}\right|^2 \mathrm{d}t \\
&= \frac{1}{2\pi}\int_0^{2\pi}\left(\sum_{n=0}^{\infty} a_n r^n e^{int}\right)\overline{\left(\sum_{n=0}^{\infty} a_n r^n e^{int}\right)}\mathrm{d}t \\
&= \frac{1}{2\pi}\int_0^{2\pi}\sum_{n=0}^{\infty} a_n r^n e^{int}\sum_{m=0}^{\infty}\overline{a_m} r^m e^{-imt}\mathrm{d}t \\
&= \frac{1}{2\pi}\int_0^{2\pi}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} a_n r^n e^{int}\overline{a_m} r^m e^{-imt}\mathrm{d}t \\
&= \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} a_n\overline{a_m}\frac{1}{2\pi}\int_0^{2\pi} r^{n+m} e^{i(n-m)t}\mathrm{d}t \\
&= \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} a_n\overline{a_m}\frac{1}{2\pi} r^{n+m}\delta_{n,m}\int_0^{2\pi} 1\mathrm{d}t \\
&= \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} a_n\overline{a_m} r^{n+m}\delta_{n,m},
\end{aligned}$$

where

$$\delta_{n,m} = \begin{cases} 1 & n = m \\ 0 & n \neq m. \end{cases}$$

The interchanges of summation and integration are justified by the uniform convergence of the series $f$ on $\partial B_r(0)$, and hence

$$I_k = \sum_{n=0}^{\infty} |a_n|^2 r^{2n}.$$

□