MA244 Analysis III Support Class - Week 8

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March 28, 2023

1 Week 8

Question 1.

Let $f : \Omega \to \mathbb{C}$ be an analytic function on an open set Ω. State the Cauchy-Riemann equations satisfied by the real and imaginary part of f and prove that both are harmonic functions. i,e. $\Delta\text{Re}(f) = 0 = \Delta\text{Im}(f)$ on Ω, where $\Omega = \partial_x^2 + \partial_y^2$ is the two-dimensional Laplacian. (Assume that all relevatn partial derivatives exist and $\partial_x\partial_y = \partial_y\partial_x$.)

Solution.

Suppose $f(z) = u(x, y) + iv(x, y)$, then Cauchy-Riemann equations are

$$u_x = v_y \qquad u_y = -v_x.$$

We calculate the Laplacian of u,

$$\Delta u = u_{xx} + u_{yy} = (v_y)_x + (-v_x)_y = 0.$$

While calculating the Laplacian of v,

$$\Delta v = v_{xx} + v_{yy} = (-u_y)_x + (u_x)_y = 0.$$

□

Question 2.

Find the harmonic conjugate for $u : (x, y) \to \sin x \cosh y$ on $\mathbb{R}^2$. That is, find $v : \mathbb{R}^2 \to \mathbb{R}$ so that the function $f(z) = u(x, y) + iv(x, y)$ for $z = x + iy$ is analytic.

Solution.

The Cauchy-Riemann equations state that

$$u_x = v_y \qquad u_y = -v_x.$$

Therefore, it tells us $v_y = \sin x \cosh y$, which implies that

$$v(x, y) = \cos x \sinh y + f(x),$$

because for v_y, we only differentiate v with respect to y, there could exists a function with x.

To find f, we use $u_y = -v_x$. Note that

$$v_x = -\sin x \sinh y + f'(x) = -u_y = -\sin x \sinh y,$$

which implies $f'(x) = 0$ and $f(x)$ is a constant. Therefore all the $v : \mathbb{R}^2 \to \mathbb{R}$ are in the form

$$v(x, y) = \cos x \sinh y + C.$$

□

Question 3.

Given a sequence $(a_n)_{n=0}^{\infty}$, define what is means for the power series

$$\sum_{n=0}^{\infty} a_n z^n$$

to have radius of convergence R. Indicate the range of values of R and a formula to calculate R in terms of the coefficients a_n.

Solution.

The radius of convergence $R \in [0, \infty]$ exists such that

$$\sum_{n=0}^{\infty} a_n z^n$$

converges for all $|z| < R$ and diverges for $|z| > R$. The formula is

$$R = \frac{1}{\lim_{n\to\infty} \sup |a_n|^{\frac{1}{n}}}.$$

□

Question 4.

Calculate the radius of convergence of the following power series. For $a \neq 0$,

$$\sum_{k=0}^{\infty} a^k z^k, \qquad \sum_{k=0}^{\infty} z^{k!}, \qquad \sum_{k=0}^{\infty} k! z^k.$$

State any convergence tests (either for series or power series) that you use.

Solution.

For the first series, use ratio tests: Let $a_n \neq 0$ for all $n \geq N$, and assume that $\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|}$ exists. Then $\sum_{n=0}^{\infty} a_n z^n$ has radius of convergence $R = \lim_{n\to\infty} \frac{|a_n|}{|a_{n+1}|}$.

Therefore,

$$R = \lim_{k\to\infty} \frac{|a^k|}{|a^{k+1}|} = |a|^{-1}.$$

For the second one, we still use the Ratio test, but this one the normal version:

Consider $\sum_{n=0}^{\infty} a_n$ and assume that $a_n \neq 0$ for all n. Then

(a) If $\limsup \frac{|a_{n+1}|}{|a_n|} < 1$, then $\sum_{n=0}^{\infty} a_n$ is convergent.

(b) If $\frac{|a_{n+1}|}{|a_n|} \geq 1$ for all $n > N$, then $\sum_{n=0}^{\infty} a_n$ converges.

Therefore,

$$\lim_{k\to\infty} \frac{|z|^{(k+1)!}}{|z|^{k!}} = \lim_{k\to\infty} |z|^{k(k!)},$$

which the limit is 0 for $|z| < 1$ and ∞ for $|z| > 1$. Therefore, the radius of convergence is 1.

For the last one, we use ratio test for the first version:

$$R = \lim_{k\to\infty} \frac{|k!|}{|(k+1)!|} = \lim_{k\to\infty} \frac{1}{k+1} = 0.$$

□