MA244 Analysis III Support Class - Week 5

Louis Li

March 27, 2023

1 Week 5

Question 1.

Consider the sequence of functions

$$h_n(x) = \frac{x}{1 + x^n}, n \in \mathbb{N},$$

on the domain $[0, \infty)$.

(a) Find the pointwise limit h of (h_n) on $[0, \infty)$.

(b) Prove that (h_n) does not converge to h uniformly on $[0, \infty)$.

(c) Prove that there exists a subset of $[0, \infty)$ over which the convergence is uniform.

Solution.

(a) Note that $\lim_{n\to\infty} h_n(x)$ has different values in different xs. We have

$$\lim_{n\to\infty} \frac{x}{1+x^n} = \begin{cases} x & 0 < x < 1 \\ \frac{1}{2} & x = 1 \\ 0 & x > 1. \end{cases}$$

We see that h has a jump discontinuity at $x = 1$:

$$\lim_{x\to 1-} h(x) = 1 \neq 0 = \lim_{x\to 1+} h(x).$$

As $h(1) = \frac{1}{2}$, h is neither right nor left continuous at $x = 1$.

(b) Suppose (h_n) converges uniformly on $[0, \infty)$. By uniform convergence and continuous theorem, if (f_n) converges uniformly to f, then f is continuous. However by (a), we know that h is not continuous at $x = 1$, therefore it is a contradiction and (h_n) does not converge uniformly on $[0, \infty)$.

(c) Clearly from part (b), we know the subset cannot contain 1. Claim that the subset is $[0, a]$ for $a < 1$ and $[b, \infty)$ for $b > 1$.

Proof: For the first subset $[0, a)$, where $a < 1$, we have

$$\sup_{[0,a]} |h_n - h| = \sup_{[0,a]} \left| \frac{x}{1+x^n} - x \right| = \sup_{[0,a]} \left| \frac{x^{n+1}}{1+x^n} \right| \leq \sup_{[0,a]}(x^{n+1}) \leq a^{n+1}.$$

As $n \to \infty$, $a^{n+1} \to 0$ since $a < 1$. Hence,

$$\sup_{[0,a]} |h_n - h| \leq 0 < \varepsilon,$$

and it converges uniformly on $[0, a]$.

For the second subset $[b, \infty)$, where $b > 1$, we have

$$\sup_{[b,\infty)} |h_n - h| = \sup_{[b,\infty)} \left| \frac{x}{1+x^n} - 0 \right| \leq \sup_{[b,\infty)} \frac{x}{x^n} \leq \frac{1}{b^{n-1}}.$$

As $n \to \infty$, $\frac{1}{b^{n-1}} \to 0$. Hence,

$$\sup_{[b,\infty)} |h_n - h| \leq 0 < \varepsilon,$$

and it also converges uniformly on $[b, \infty)$.

□

Question 2.

Decide which of the following statements are true and which are false. Present a proof or a counterexample to support your answer.

(a) If $f_n \to f$ pointwise on a closed bounded set K, then $f_n \to f$ uniformly on K.

(b) If $f_n \to f$ uniformly on A and g is a bounded function on A, then $f_n g \to fg$ uniformly on A.

(c) If $f_n \to f$ uniformly on A, and if each f_n is bounded on A, then f must also be bounded.

(d) If $f_n \to f$ uniformly on a set A, and if $f_n \to f$ uniformly on a set B, then $f_n \to f$ uniformly on $A \cup B$.

(e) If $f_n \rightrightarrows f$ uniformly on an interval, and if each f_n is increasing, then f is also increasing.

(f) If $f_n \to f$ pointwise on an interval, and if each f_n is increasing, then f is also increasing.

Solution.

(a) False. Consider the counterexample

$$f_n(x) = \begin{cases} 2nx & x \in \left[0, \frac{1}{2n}\right) \\ -2n\left(x - \frac{1}{n}\right) & x \in \left[\frac{1}{2n}, \frac{1}{n}\right) \\ 0 & x \in \left[\frac{1}{n}, 1\right]. \end{cases}$$

We see that $f_n(x) \to f = 0$. However for every n, we have $f_n\left(\frac{1}{2n}\right) = 1$ and therefore

$$\sup_{[0,1]} |f_n(x) - f| = \sup_{[0,1]} |f_n(x)| = 1,$$

which is not smaller than ε.

(b) True.

Proof: Since g is a bounded function on A, then

$$\sup_{x\in A} |g| \leq M,$$

for some M and since $f_n \to f$ uniformly on A, then

$$\sup_{x\in A} |f_n - f| < \varepsilon,$$

for all $\varepsilon > 0$. Therefore,

$$\sup_{x\in A} |f_n g - fg| = \sup_{x\in A} |f_n - f|\,|g| = \sup_{x\in A} |f_n - f| \sup_{x\in A} |g| < M\varepsilon.$$

(c) True.

Proof: Note that

$$|f| = |f - f_n + f_n| \leq |f - f_n| + |f_n|.$$

Since $f_n \to f$ uniformly on A and f_n is bounded, then

$$\sup_{x\in A} |f - f_n| < \varepsilon,$$

and

$$\sup_{x\in A} |f_n| < M.$$

Therefore,

$$|f| \leq M + 1$$

since $\varepsilon > 0$, and that shows f is bounded.

(d) True.

Proof: Since $f_n \to$ f uniformly on A and B, then

$$\sup_{x\in A} |f_n - f| < \varepsilon_1 \qquad \sup_{x\in B} |f_n - f| < \varepsilon_2.$$

Choose N to be the largest among all Ns in A and B, so for all $n > N$, we have

$$\sup_{x\in A\cup B} |f_n - f| < \varepsilon_3 < \varepsilon.$$

Therefore, $f_n \to f$ uniformly on $A \cup B$.

(e) True.

Proof: Assume that f is not increasing, that means there exists a $x < y$, such that $f(x) > f(y)$. Since $f_n \rightrightarrows f$ uniformly, then there exists N such that

$$|f(x) - f_n(x)| < \varepsilon \qquad |f(y) - f_n(y)| < \varepsilon$$

for all $n \geq N$. The above also can be rewritten as

$$f(x) - \varepsilon < f_n(x) < f(x) + \varepsilon \qquad \text{and} \qquad f(y) - \varepsilon < f_n(y) < f(y) + \varepsilon.$$

Choose $\varepsilon = \frac{f(x)-f(y)}{2}$, we have

$$f(y) + \varepsilon = f(x) - \varepsilon < f_n(x),$$

and since f_n is increasing, we have $f_n(x) < f_n(y)$, but that means

$$f(y) + \varepsilon < f_n(x) < f_n(y)$$

which contradicts the fact that $f_n(y) < f(y)+\varepsilon$. Therefore the assumption that f is not increasing is false and the statement should be true.

(f) True.

Proof: Since uniform convergence implies pointwise convergence and part (e) is true, then part (f) is true.

□

Question 3.

Show that if f_n and g_n are bounded and converge uniformly to f and g respectively, then $f_n g_n$ converges uniformly to fg.

Proof.

Since f_n and g_n are bounded and converge uniformly to f and g, then we have

$$|f| = |f_n + f - f_n| \leq |f_n| + |f - f_n| \leq M + \varepsilon,$$

for some $M, \varepsilon > 0$. Similar for g, and that shows f and g are bounded. Also we have

$$\begin{aligned} |f_n g_n - fg| &= |f_n g_n - f_n g + f_n g - fg| \\ &\leq |f_n| \, |g_n - g| + |g| \, |f_n - f| \\ &\leq M_f \, |g_n - g| + M_g \, |f_n - f| \, . \end{aligned}$$

Since f_n, g_n converge uniformly, for any $\varepsilon > 0$, we can choose N large enough such that

$$|g_n - g| \leq \frac{\varepsilon}{2M_f} \text{ and } |f_n - f| \leq \frac{\varepsilon}{2M_g}.$$

Therefore,

$$|f_n g_n - fg| \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Since that works for every x, then

$$\sup |f_n g_n - fg| \leq \varepsilon$$

and that implies $f_n g_n \rightrightarrows fg$. □