MA244 Analysis III Support Class - Week 7

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1 Week 7

Question 1.

Let M be a strictly positive integer. Show that the series

$$\sum_{n=M}^{\infty} \frac{1}{(x-n)^2} \quad \text{and} \quad \sum_{n=M}^{\infty} \frac{1}{(x+n)^2}$$

converge uniformly for $|x| \leq \frac{M}{2}$. Conclude that the series

$$\sum_{n=M}^{\infty} \left(\frac{1}{(x-n)^2} + \frac{1}{(x+n)^2} \right)$$

converges uniformly on $\left[-\frac{M}{2}, \frac{M}{2}\right]$. Let the limit be f_M. Prove that f_M is continuous on $\left[-\frac{M}{2}, \frac{M}{2}\right]$ and differentiable on $\left(-\frac{M}{2}, \frac{M}{2}\right)$.

Proof.

Note that

$$\sum_{n=M}^{\infty} \frac{1}{(n-x)^2} \leq \sum_{n=M}^{\infty} \frac{1}{(n-\frac{M}{2})^2} \leq \sum_{n=M}^{\infty} \frac{1}{n^2} < \infty$$

and similarly for $\displaystyle\sum_{n=M}^{\infty} \frac{1}{(x+n)^2}$. Hence by Weierstrass M-test, both series converge uniformly for $|x| \leq \frac{M}{2}$.

Define $h_n(x) = \frac{1}{(x-n)^2} + \frac{1}{(x+n)^2}$, and it is a sequence of C^1 functions on $\left[-\frac{M}{2}, \frac{M}{2}\right]$. Hence, the limit f_M is continuous as the convergence is uniform. Compute $h_n'(x)$, we see it is

$$h_n'(x) = -\frac{2}{(x-n)^3} - \frac{2}{(x+n)^3}.$$

Note that

$$\left| -\frac{2}{(x-n)^3} - \frac{2}{(x+n)^3} \right| \leq \frac{4}{(n-\frac{M}{2})^3}$$

and $\displaystyle\sum_{n=M}^{\infty} \frac{4}{(n-\frac{M}{2})^3}$ converges, hence by Weierstrass M-test, $\displaystyle\sum_{n=M}^{\infty} h_n'(x)$ converges uniformly on $\left[-\frac{M}{2}, \frac{M}{2}\right]$. Then by the continuity and uniformly convergence theorem, f_M is C^1.

□

Question 2.

Use the result of part (a) to show that the function

$$F(x) = \sum_{n=-\infty}^{\infty} \frac{1}{(x-n)^2}$$

is well defined, continuous and differentiable on $\mathbb{R} \setminus \mathbb{Z}$. Hint: Use an appropriate (possibly x-dependent) decomposition of the above series as the sum of two series from part (a). Show that $F(x+1) = F(x)$ for all $x \in \mathbb{R} \setminus \mathbb{Z}$.

Proof.

Consider the hint, we separate $F(x)$ into a sum of two series. For $k \geq 0$, we have

$$F(x) = \sum_{n=-\infty}^{\infty} \frac{1}{(x-n)^2} = \sum_{p=-2(k+1)}^{2(k+1)} \frac{1}{(x-p)^2} + \sum_{p=2k+3}^{\infty} \frac{1}{(x-p)^2} + \sum_{p=2k+3}^{\infty} \frac{1}{(x+p)^2}.$$

We know that $\sum_{p=-2(k+1)}^{2(k+1)}$ is continuously differentiable on $\mathbb{R} \setminus \mathbb{Z}$, while the last two sums are also continuously differentiable on $\left[-\frac{2k+3}{2}, \frac{2k+3}{2}\right]$ by Question 1. Hence for $k \geq 0$, $F(x)$ is continuously differentiable. Similar arguments follow for $k \leq 0$, and therefore, $F(x)$ is well defined, continuous and differentiable on $\mathbb{R} \setminus \mathbb{Z}$. For any $N \in \mathbb{Z}$, $x \in \mathbb{R} \setminus \mathbb{Z}$, we have

$$\begin{aligned}
F(x+1) &= \lim_{N_1 \to -\infty} \sum_{n=N_1}^{N} \frac{1}{(x+1-n)^2} + \lim_{N_2 \to \infty} \sum_{n=N+1}^{N_2} \frac{1}{(x+1-n)^2} \\
&= \lim_{N_1 \to -\infty} \sum_{n=N_1-1}^{N-1} \frac{1}{(x-n)^2} + \lim_{N_2 \to \infty} \sum_{n=N}^{N_2-1} \frac{1}{(x-n)^2} \\
&= \lim_{N_1 \to -\infty} \sum_{n=N_1}^{N-1} \frac{1}{(x-n)^2} + \lim_{N_2 \to \infty} \sum_{n=N}^{N_2} \frac{1}{(x-n)^2} \\
&= \sum_{n=-\infty}^{N-1} \frac{1}{(x-n)^2} + \sum_{n=N}^{\infty} \frac{1}{(x-n)^2} \\
&= \sum_{n=-\infty}^{\infty} \frac{1}{(x-n)^2} \\
&= F(x).
\end{aligned}$$

□

Question 3.

(a) Let $g : \mathbb{R} \to \mathbb{R}$ be a continuous functions: $g(x) = g(x+1)$ for all $x \in \mathbb{R}$. Prove that g is bounded.

(b) Let f be a bounded function on $\mathbb{R}$ such that

$$f(x) = \frac{1}{4}\left(f\left(\frac{x}{2}\right) + f\left(\frac{x+1}{2}\right)\right)$$

for all x. Prove that $f(x) = 0$ for all x.

Proof.

(a) Since g is continuous on $\mathbb{R}$, then it must be continuous on $[0,1]$. Hence by the boundedness theorem, g must be bounded on $[0,1]$, i.e. there exists a $M \geq 0$ such that $|g(x)| \leq M$. Since $g(x) = g(x+1)$ for all $x \in \mathbb{R}$, that means the function $g(x)$ has a period of 1. i.e. for every $y \in \mathbb{R}$, it can be represented as $I + r$, where $I \in \mathbb{Z}$ and $r \in [0,1)$. Hence,

$$|g(y)| = |g(I + r)| = |g(r)| \leq M.$$

(b) Since f is bounded on $\mathbb{R}$, then there exists a $M \geq 0$, such that $|f(x)| \leq M$. Note that

$$|f(x)| = \frac{1}{4}\left(\left|f\left(\frac{x}{2}\right)\right| + \left|f\left(\frac{x+1}{2}\right)\right|\right) \leq \frac{1}{4}(M + M) = \frac{1}{2}M,$$

which is true for every $x \in \mathbb{R}$.

Consider $f\left(\frac{x}{2}\right)$ and $f\left(\frac{x+1}{2}\right)$, we have

$$f\left(\frac{x}{2}\right) = \frac{1}{4}\left(f\left(\frac{x}{4}\right) + f\left(\frac{x+2}{4}\right)\right) \quad \text{and} \quad f\left(\frac{x+1}{2}\right) = \frac{1}{4}\left(f\left(\frac{x+1}{4}\right) + f\left(\frac{x+3}{4}\right)\right).$$

Similarly, we have

$$f\left(\frac{x}{2}\right) \leq \frac{1}{2}M \quad \text{and} \quad f\left(\frac{x+1}{2}\right) \leq \frac{1}{2}M.$$

Substitute back to $|f(x)|$, we get

$$|f(x)| = \frac{1}{4}\left(\left|f\left(\frac{x}{2}\right)\right| + \left|f\left(\frac{x+1}{2}\right)\right|\right) \leq \frac{1}{4}\left(\frac{1}{2}M + \frac{1}{2}M\right) = \frac{1}{4}M.$$

If we continue n steps, we get

$$|f(x)| \leq \frac{1}{2^n}M.$$

For every $x \in \mathbb{R}$, we see that

$$\lim_{n\to\infty} |f(x)| \leq \lim_{n\to\infty} \frac{1}{2^n}M = 0.$$

Hence, $f(x) \equiv 0$.

□

Question 4.

(a) Show by L'Hopital's rule that $\pi^2 \csc^2(\pi x) - \frac{1}{x^2} \to \frac{\pi^2}{3}$ as $x \to 0$.

(b) Let F, f_M be functions defined in part (b) and (a) respectively. Let $g : \mathbb{R} \to \mathbb{R}$ be defined as follows:

$$g(x) = F(x) - \pi^2 \csc^2(\pi x) = f_1(x) - \left(\pi^2 \csc^2(\pi x) - \frac{1}{x^2}\right),\ x \in \mathbb{R} \setminus \mathbb{Z},$$

$g(x) = f_1(0) - \frac{\pi^2}{3}, x \in \mathbb{Z}$. Prove that g is continuous on $\mathbb{R}$ and $g(x) = g(x+1)$ for all $x \in \mathbb{R}$.

Proof.

(a) By L'Hopital's rule,

$$\begin{aligned}
\lim_{x\to 0}\left(\pi^2\csc^2(\pi x)-\frac{1}{x^2}\right) &= \lim_{x\to 0}\frac{x^2\pi^2\csc^2(\pi x)-1}{x^2}\\
&= \lim_{x\to 0}\frac{x^2\pi^2-\sin^2(\pi x)}{x^2\sin^2(\pi x)}\\
&= \lim_{x\to 0}\frac{2\pi^2 x-2\pi\sin(\pi x)\cos(\pi x)}{2\pi x^2\sin(\pi x)\cos(\pi x)+2x\sin^2(\pi x)}\\
&= \lim_{x\to 0}\frac{\pi^2 x-\pi\sin(\pi x)\cos(\pi x)}{\pi x^2\sin(\pi x)\cos(\pi x)+x\sin^2(\pi x)}\\
&= \lim_{x\to 0}\frac{\pi^2 x-\frac{\pi}{2}\sin(2\pi x)}{\frac{\pi}{2}x^2\sin(2\pi x)+x\sin^2(\pi x)}\\
&= \lim_{x\to 0}\frac{\pi^2-\pi^2\cos(2\pi x)}{\pi^2x^2\cos(2\pi x)+2\pi x\sin(2\pi x)+\sin^2(\pi x)}\\
&= \lim_{x\to 0}\frac{2\pi^2\sin(2\pi x)}{-2\pi^2x^2\sin(2\pi x)+6\pi x\cos(2\pi x)+3\sin(2\pi x)}\\
&= \frac{4\pi^3}{12\pi}\\
&= \frac{\pi^2}{3}.
\end{aligned}$$

Note: Have to use L'Hopital's rule for 4 times.

(b) Since F and $\pi^2\csc^2(\pi x)$ are continuous, then $g(x)$ is continuous on $\mathbb{R}\setminus\mathbb{Z}$. It is also continuous at $x=0$: note that f_1 is continuous on $\left[-\frac{1}{2},\frac{1}{2}\right]$. Therefore,

$$\lim_{x\to 0} g(x) = f_1(0) - \lim_{x\to 0}\left(\pi^2\csc^2(\pi x)-\frac{1}{x^2}\right) = f_1(0)-\frac{\pi^2}{3} = g(0).$$

Since F is periodic with period 1 on $\mathbb{R}\setminus\mathbb{Z}$, and g is continuous at every $x=n$, where $n\in\mathbb{N}$:

$$\lim_{x\to n} g(x) = \lim_{x\to 0} g(x+n) = \lim_{x\to 0} g(x) = g(0) = g(n).$$

So g is continuous on $\mathbb{R}$. It is periodic on $\mathbb{R}\setminus\mathbb{Z}$ and $g(n)=g(0)$. Hence, $g(x)=g(x+1)$ for all $x\in\mathbb{R}$.

□

Question 5.

Bringing everything together: consider function g defined in part (d).

(a) Prove that it is bounded and satisfies the equation of part (c). Thus prove that $g\equiv 0$ and so

$$\sum_{n=-\infty}^{\infty}\frac{1}{(x-n)^2} = \pi^2\csc^2(\pi x)$$

for all $x\in\mathbb{R}\setminus\mathbb{Z}$.

(b) Show that

$$\sum_{n=1}^{\infty} n^{-2} = \frac{\pi^2}{6}.$$

Proof.

(a) Note that the function g defined in part (d) is periodic and continuous. Therefore by Question 3, it is bounded. The equation in Question 4 is

$$f(x) = \frac{1}{4}\left(f\left(\frac{x}{2}\right) + f\left(\frac{x+1}{2}\right)\right),$$

and substitute it with $F(x)$ we see that

$$\begin{aligned}
\frac{1}{4}\left(F\left(\frac{x}{2}\right) + F\left(\frac{x+1}{2}\right)\right) &= \frac{1}{4}\left(\sum_{n=M}^{\infty} \frac{1}{(\frac{x}{2}-n)^2} + \sum_{n=M}^{\infty} \frac{1}{(\frac{x+1}{2}-n)^2}\right) \\
&= \frac{1}{4}\left(\sum_{n=M}^{\infty} \frac{4}{(x-2n)^2} + \sum_{n=M}^{\infty} \frac{4}{(x-2n+1)^2}\right) \\
&= \left(\sum_{n\in 2\mathbb{Z}} + \sum_{n\in 2\mathbb{Z}+1}\right)\frac{1}{(x-n)^2} \\
&= F(x).
\end{aligned}$$

Note that

$$\begin{aligned}
\frac{1}{4}\left(\csc^2\left(\frac{\pi x}{2}\right) + \csc^2\left(\frac{\pi(x+1)}{2}\right)\right) &= \frac{1}{4}\left(\csc^2\left(\frac{\pi x}{2}\right) + \frac{1}{\cos^2\left(\frac{\pi x}{2}\right)}\right) \\
&= \frac{1}{4\sin^2\left(\frac{\pi x}{2}\right)\cos^2\left(\frac{\pi x}{2}\right)} \\
&= \frac{1}{\sin^2\left(\frac{\pi x}{2}\right)} \\
&= \csc^2\left(\frac{\pi x}{2}\right).
\end{aligned}$$

Therefore, we see that both $\csc^2(\pi x)$ and $F(x)$ satisfy the equation in Question 3. By linearity, $g(x) = F(x) - \pi^2\csc^2(\pi x)$ also satisfies the equation on $\mathbb{R}\setminus\mathbb{Z}$. Follow the same step in Question 3, since g is bounded on $\mathbb{R}$ and by taking the limit of $x \to n$ to prove the equation in Question 3 is satisfied by g for any $x \in \mathbb{R}$, we conclude $g(x) \equiv 0$ by Question 3. Hence it means

$$\begin{aligned}
F(x) &= \pi^2\csc^2(\pi x) \\
\sum_{n=-\infty}^{\infty} \frac{1}{(x-n)^2} &= \pi^2\csc^2(\pi x)
\end{aligned}$$

for all $x \in \mathbb{R}\setminus\mathbb{Z}$.

(b) We can write part (a) into

$$\begin{aligned}
\sum_{n=-\infty}^{-1} \frac{1}{(x-n)^2} + \frac{1}{x^2} + \sum_{n=1}^{\infty} \frac{1}{(x-n)^2} &= \pi^2\csc^2(\pi x) \\
2\sum_{n=1}^{\infty} \frac{1}{(x-n)^2} &= \pi^2\csc^2(\pi x) - \frac{1}{x^2}.
\end{aligned}$$

Hence taking the limit we have

$$\begin{aligned}
\lim_{x\to 0} 2\sum_{n=1}^{\infty} \frac{1}{(x-n)^2} &= \lim_{x\to 0}\left(\pi^2\csc^2(\pi x) - \frac{1}{x^2}\right)\\
2\sum_{n=1}^{\infty} n^{-2} &= \frac{\pi^2}{3}\\
\sum_{n=1}^{\infty} n^{-2} &= \frac{\pi^2}{6}.
\end{aligned}$$

$\square$