MA244 Analysis III Support Class - Week 6

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1 Week 6

Question 1.

Consider the sequence of functions

$$g_n(x) = \frac{x^n}{n},\ x \in [0,1],\ n \in \mathbb{N}.$$

(a) Does $g_n(x)$ converge pointwise for $x \in [0,1]$? Does it converge uniformly? Determine the limit, $g(x)$, if it exists, and differentiate it to find $g'(x)$.

(b) Does (g_n') converge pointwise and/or uniformly on $[0,1]$? If $\lim_{n\to\infty} g_n' = h$, then is h equal to g'?

Solution.

(a) g_n(x) converge pointwise for $x \in [0,1]$. We see that for $x \in [0,1]$,

$$0 \le \frac{x^n}{n} \le \frac{1}{n},$$

and

$$\lim_{n\to\infty} 0 \le \lim_{n\to\infty} \frac{x^n}{n} \le \lim_{n\to\infty} \frac{1}{n}.$$

By Squeeze theorem, we have

$$\lim_{n\to\infty} \frac{x^n}{n} = 0,$$

for $x \in [0,1]$.

It converges uniformly to $g(x) = 0$ also. Choose $n > N > \frac{1}{\varepsilon}$, we have

$$|g_n(x) - g(x)| = \left|\frac{x^n}{n}\right| \le \left|\frac{1}{n}\right| < \varepsilon.$$

Therefore, $g'(x) = 0$.

(b) First of all we get $(g_n') = x^{n-1}$. It still converges pointwise to $h : [0,1] \to \mathbb{R}$ as

$$\lim_{n\to\infty} x^{n-1} = 0, \text{ where } 0 \le x < 1,$$

and $h(1) = 1$. Therefore h is discontinuous at $x = 1$ and

$$\lim_{x\to 1-} h(x) = 0 \neq 1 = h(1).$$

Therefore, $g_n' \to h$ but $g_n' \not\rightrightarrows h$ and $g' \neq h$.

□

Question 2.

(a) Show that

$$g(x) = \sum_{n=1}^{\infty} \frac{\cos(2^n x)}{2^n}$$

is continuous on $\mathbb{R}$.

(b) Prove that

$$h(x) = \sum_{n=1}^{\infty} \frac{x^n}{n^2}$$

is continuous on $[-1, 1]$.

Solution.

(a) Note that

$$\left| \frac{\cos(2^n x)}{2^n} \right| \leq \frac{1}{2^n}, \quad x \in \mathbb{R},$$

and that

$$\sum_{n=1}^{\infty} \frac{1}{2^n} < \infty,$$

therefore it converges and by Weierstrass M-test, $g(x)$ converges uniformly. Since uniform convergence implies continuity in this case as each term in the sum is continuous, then $g(x)$ is continuous on $\mathbb{R}$.

(b) Similarly, note that

$$\left| \frac{x^n}{n^2} \right| \leq \frac{1}{n^2}, \quad x \in [-1, 1],$$

and that

$$\sum_{n=1}^{\infty} \frac{1}{n^2} < \infty,$$

therefore it converges and by Weierstrass M-test, $h(x)$ converges uniformly on $x \in [-1, 1]$. Since uniform convergence implies continuity in this case as each term in the sum is continuous, then $h(x)$ is continuous on $[-1, 1]$.

□

Question 3.

Let

$$h(x) = \sum_{n=1}^{\infty} \frac{1}{x^2 + n^2}.$$

(a) Show that h is a continuous function on $\mathbb{R}$.

(b) Is h differentiable? If so, is the derivative function h' continuous?

Proof.

(a) Note that

$$\left|\frac{1}{x^2+n^2}\right| \leq \frac{1}{n^2}, \quad x \in \mathbb{R},$$

and

$$\sum_{n=1}^{\infty} \frac{1}{n^2} < \infty,$$

therefore it converges and by Weierstrass M-test, $h(x)$ converges uniformly. Since uniform convergence implies continuity in this case as each term in the sum is continuous, then h is a continuous function on $\mathbb{R}$.

(b) Define $h_k(x) = \sum_{n=1}^{k} \frac{1}{x^2+n^2}$, $k \geq 1$. This is a sequence of C^1 functions on $\mathbb{R}$. Hence h is differentiable. To prove that the limit is C^1, we show that the sequence (h_k') converges uniformly and we will prove that it is uniformly Cauchy:

$$h_k'(x) = \sum_{n=1}^{k} \frac{-2x}{(x^2+n^2)^2}.$$

By Cauchy-Schwarz equality, $2|x|n \leq x^2+n^2$, where $n \geq 1$, we have

$$\left|h_k'(x)\right| \leq \sum_{n=1}^{k} \frac{2|x|}{(x^2+n^2)^2} \leq 2\sum_{n=1}^{k} \frac{x^2+n^2}{(x^2+n^2)^2} \leq 2\sum_{n=1}^{k} \frac{1}{n^2}.$$

Therefore if $k < l$, we have

$$\left|h_l'(x) - h_k'(x)\right| \leq 2 \sum_{n=k+1}^{l} \frac{1}{n^2}.$$

Given $\varepsilon > 0$, choosing N such that $\sum_{n=N}^{\infty} \frac{1}{n^2} < \varepsilon$, and $|h_l'(x) - h_k'(x)| < \varepsilon$.

Hence, (h_k') is uniformly Cauchy and in turn it is uniformly convergent and hence h' is C^1.

□

Question 4.

Let (f_n) be a sequence of continuously differentiable functions defined on the closed interval $[a,b]$ and assume that (f_n') converges uniformly on $[a,b]$. Show that if there exists a point $x_0 \in [a,b]$ where $f_n(x_0)$ is convergent, then (f_n) converges uniformly on $[a,b]$.

Solution.

We will show its uniform convergence by showing (f_n) is uniformly Cauchy. We have

$$f_n(x) - f_m(x) = f_n(x) - f_m(x) - (f_n(x_0) - f_m(x_0)) + (f_n(x_0) - f_m(x_0))$$

□

and hence

$$|f_n(x) - f_m(x)| \leq |f_n(x) - f_m(x) - (f_n(x_0) - f_m(x_0))| + |f_n(x_0) - f_m(x_0)|.$$

By Mean Value Theorem on $f_n(x) - f_m(x)$, we have

$$f_n'(\tau_{n,m}) - f_m'(\tau_{n,m}) = \frac{f_n(x) - f_m(x) - (f_n(x_0) - f_m(x_0))}{x - x_0}$$

for some $\tau_{n,m} \in [a, b]$ that depends on x and x_0. Therefore,

$$|f_n(x) - f_m(x)| \leq |x - x_0| \left\| f_n' - f_m' \right\|_\infty + |f_n(x_0) - f_m(x_0)| \leq |b - a| \left\| f_n' - f_m' \right\|_\infty + |f_n(x_0) - f_m(x_0)| \, .$$

As (f_n') converges uniformly on $[a, b]$, therefore given $\varepsilon > 0$, there exists N such that

$$\left\| f_n' - f_m' \right\|_\infty < \frac{\varepsilon}{2\,|b - a|}$$

and

$$|f_n(x_0) - f_m(x_0)| < \frac{\varepsilon}{2}.$$

Therefore,

$$|f_n(x) - f_m(x)| \leq |b - a| \frac{\varepsilon}{2\,|b - a|} + \frac{\varepsilon}{2} = \varepsilon,$$

and (f_n) converges uniformly on $[a, b]$.