# MA244 Analysis III Support Class - Week 4

Louis Li

March 24, 2023

## 1 Week 4

**Question 1.**

Compute the derivative of the functions

$$f(x) = \int_0^{x^2} \sin(s^3)\mathrm{d}s \qquad g(x) = \int_{x^3}^{2+x^6} e^{-s^2}\mathrm{d}s$$

*Solution.*

Let $u(s) = \int \sin(s^3)\mathrm{d}s$. Then $u'(s) = \sin(s^3)$ and $f(x) = u(x^2) - u(0)$.

Therefore,

$$\begin{aligned} f'(x) &= 2xu'(x^2) - u'(0) \\ &= 2x\sin(x^6). \end{aligned}$$

Similarly, let $v(s) = \int e^{-s^2}\mathrm{d}s$. Then $v'(s) = e^{-s^2}$ and $g(x) = v(2+x^6) - v(x^3)$.

Therefore,

$$\begin{aligned} g'(x) &= 6x^5v'(2+x^6) - 3x^2v'(x^3) \\ &= 6x^5e^{-(2+x^6)^2} - 3x^2e^{-x^6}. \end{aligned}$$

□

**Question 2.**

Consider the integrability of $f(x) = \frac{1}{x}$ on $[-1,1]\setminus\{0\}$. Show that for any $s \in \mathbb{R}$ there exist non-negative functions $a(\varepsilon), b(\varepsilon)$ that tend to 0 as $\varepsilon$ tends to zero and for which

$$\lim_{\varepsilon\to 0+}\left[\int_{-1}^{-a(\varepsilon)} \frac{\mathrm{d}x}{x} + \int_{b(\varepsilon)}^{1} \frac{\mathrm{d}x}{x}\right] = s.$$

This result proves that the improper integral $\int_{-1}^{1} x^{-1}\mathrm{d}x$ does not exist. Define the integral in the sense of the principal value: for $f : [-1,1]\setminus\{0\}$,

$$P.V. \int_{-1}^{1} f(x)\mathrm{d}x = \lim_{\varepsilon\to 0+}\left[\int_{-1}^{-\varepsilon} f(x)\mathrm{d}x + \int_{\varepsilon}^{1} f(x)\mathrm{d}x\right].$$

Does $P.V. \int_{-1}^{1} x^{-1}\mathrm{d}x$ exist? If it does, what is its value?

*Proof.*

Choose $a(\varepsilon) = a_0\varepsilon$ and $b(\varepsilon) = b_0\varepsilon$, where $a_0, b_0 > 0$.

Then, the limit equation becomes

$$\begin{aligned}
\lim_{\varepsilon\to 0+}\left[\int_{-1}^{-a_0\varepsilon}\frac{1}{x}\mathrm{d}x + \int_{b_0\varepsilon}^{1}\frac{\mathrm{d}x}{x}\right] &= \lim_{\varepsilon\to 0+}\left[\ln(-a_0\varepsilon) - \ln(-1) + \ln(1) - \ln(b_0\varepsilon)\right] \\
&= \lim_{\varepsilon\to 0+}\left[\ln\left(\frac{-a_0\varepsilon}{-1}\right) - \ln(b_0\varepsilon)\right] \\
&= \lim_{\varepsilon\to 0+}\left(\ln\left(\frac{a_0\varepsilon}{b_0\varepsilon}\right)\right) \\
&= \ln\left(\frac{a_0}{b_0}\right) = s.
\end{aligned}$$

Based on the definition of principal value, we see that if we let $a_0 = b_0 = 1$, then $P.V. \int_{-1}^{1} x^{-1}\mathrm{d}x$ exists, and it equals to $\ln\left(\frac{1}{1}\right) = 0$.

□

**Question 3.**

Let $f, g : [a, \infty) \to [0, \infty)$ be continuous functions such that

$$\lim_{x\to\infty}\frac{f(x)}{g(x)} = L,$$

with $0 < L < \infty$. Prove that

$$\int_a^\infty f(x)\mathrm{d}x \qquad \int_a^\infty g(x)\mathrm{d}x$$

both converge or both diverge.

Notice that the result is false if we allow $f, g : [a, b] \to \mathbb{R}$, rather than considering non-negative functions only. A counterexample is given by

$$f(x) = \frac{\sin(x)}{x} \qquad g(x) = \frac{\sin(x)}{x} + \frac{\sin^2(x)}{x\ln x}$$

where $L = 1$, whereas $f$ is integrable but $g$ is not.

Use this result or a direct comparison to decide whether the following functions are integrable on $[1, \infty)$:

$$(a)\frac{x}{x^2 + e^{-x^2}}, \quad (b)\frac{1}{x^2 + \tanh(x)}, \quad (c)\frac{1}{1 + \log(x)}.$$

*Proof.*

As the functions are continuous, they are both integrable on any bounded domain $[a, y]$.

Consider the integrals over the interval $[y, z]$ where $z$ will be sent to infinity. Hence $\lim_{x\to\infty}\frac{f(x)}{g(x)} = L$ means that for any $\varepsilon > 0$ sufficiently small, there exists $y > a$ such that

$$0 < L - \varepsilon \le \frac{f(x)}{g(x)} \le L + \varepsilon, \forall x \ge y.$$

By monotonicity of the integral,

$$(L-\varepsilon)\int_y^z g(x) \leq \int_y^z f(x) \leq (L+\varepsilon)\int_y^z g(x).$$

Similarly,

$$(L+\varepsilon)^{-1}\int_y^z f(x) \leq \int_y^z g(x) \leq (L-\varepsilon)^{-1}\int_y^z f(x).$$

The above two inequalities imply

$$\lim_{z\to\infty}\int_y^z g \leq \infty \Leftrightarrow \lim_{z\to\infty}\int_y^z f \leq \infty,$$

and we conclude that they either both converge or both diverge.

Next, the most important thing for us is to find a continuous $g : [1,\infty) \to [0,\infty)$. Consider for $n \neq 1$,

$$\int_1^\infty \frac{1}{x^n}\mathrm{d}x = \lim_{b\to\infty}\left[\frac{x^{-n+1}}{-n+1}\right]_1^b,$$

which is finite. However if $n = 1$, then $\int_1^\infty \frac{1}{x}\mathrm{d}x$ diverges. Therefore,

(a) Check

$$\lim_{x\to\infty}\frac{\frac{x}{x^2+e^{-x^2}}}{\frac{1}{x}} = \lim_{x\to\infty}\frac{x^2}{x^2+e^{-x^2}} = 1,$$

since $\int_1^\infty \frac{1}{x}\mathrm{d}x$ diverges, so does $\frac{x}{x^2+e^{-x^2}}$.

(b) Check

$$\lim_{x\to\infty}\frac{\frac{1}{x^2+\tanh(x)}}{\frac{1}{x^2}} = \lim_{x\to\infty}\frac{x^2}{x^2+\tanh(x)} = 1,$$

and $\int_1^\infty \frac{1}{x^2}\mathrm{d}x$ converges, hence $\frac{1}{x^2+\tanh(x)}$ is integrable.

(c) Using the inequality

$$\log(x) \leq x - 1$$

for $x \geq 1$,

$$\int_1^\infty \frac{1}{1+\log(x)}\mathrm{d}x \geq \int_1^\infty \frac{1}{x}\mathrm{d}x = \infty,$$

so $\int_1^\infty \frac{1}{1+\log(x)}$ diverges.

□

**Question 4.**

Show that

$$\int_0^\infty t^n e^{-t}\mathrm{d}t = n!$$

by considering the function $I_n(x) = \int_0^x t^n e^{-t}\mathrm{d}t$ and showing that

$$I_n(x) = -x^n e^{-x} + nI_{n-1}(x).$$

*Proof.*

Integrating by parts, let $u = t^n$, $\mathrm{d}v = e^{-t}\mathrm{d}t$, then $v = -e^{-t}$.

Then we have

$$\begin{aligned} I &= [uv]_0^\infty - \int_0^\infty v\mathrm{d}u \\ &= \left[-t^n e^{-t}\right]_0^\infty - \int_0^\infty -e^{-t} n t^{n-1}\mathrm{d}t \\ &= n\int_0^\infty e^{-t} t^{n-1}\mathrm{d}t. \end{aligned}$$

If we continue to use integrating by parts, we get

$$I = n!.$$

Follow the same process as above:

Let $u = t^n$, $\mathrm{d}v = e^{-t}\mathrm{d}t \implies v = -e^{-t}$.

Then

$$\begin{aligned} I_n(x) &= \int_0^x t^n e^{-t}\mathrm{d}t \\ &= [u, v]_0^x - \int_0^x v\mathrm{d}u \\ &= \left(-t^n e^{-t}\right)_0^x - \int_0^x -e^{-t} n t^{n-1}\mathrm{d}t \\ &= -x^n e^{-x} + nI_{n-1}(x). \end{aligned}$$

□