MA244 Analysis III Support Class - Week 3

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1 Week 3

Question 1.

Let $f, g : \Omega \subset \mathbb{R} \to \mathbb{R}$ be uniformly continuous and bounded. Prove that the product $fg : x \in \Omega \to f(x)g(x)$ is also uniformly continuous and bounded.

Proof.

First of all we prove the boundedness. Since f is bounded, then there exists a $M_1 > 0$, such that

$$|f| \leq M_1.$$

Similarly, there exists $M_2 > 0$ such that

$$|g| \leq M_2.$$

Hence,

$$|fg| \leq M_1 M_2 \leq M,$$

where $M = M_1 M_2 > 0$.

Next, we prove uniform continuity.

Since f is uniformly continuous, then for every $\varepsilon > 0$, there exists $\delta = \delta(\varepsilon) > 0$ such that

$$x, y \in \Omega \quad \text{and} \quad |x - y| < \delta \implies |f(x) - f(y)| < \frac{\varepsilon}{2M_1}.$$

Similarly,

$$x, y \in \Omega \quad \text{and} \quad |x - y| < \delta \implies |g(x) - g(y)| < \frac{\varepsilon}{2M_2}.$$

Thus, we have

$$\begin{aligned}
|f(x)g(x) - f(y)g(y)| &= |f(x)g(x) - f(x)g(y) + f(x)g(y) - f(y)g(y)| \\
&= |f(x)(g(x) - g(y)) + g(y)(f(x) - f(y))| \\
&\leq |f(x)(g(x) - g(y))| + |g(y)(f(x) - f(y))| \\
&= |f(x)|\,|g(x) - g(y)| + |g(y)|\,|f(x) - f(y)| \\
&\leq M_1 \frac{\varepsilon}{2M_1} + M_2 \frac{\varepsilon}{2M_2} \\
&= \varepsilon.
\end{aligned}$$

Therefore, $fg : x \in \Omega \to f(x)g(x)$ is also uniformly continuous and bounded.

□

Question 2.

(a) Let $f : [a, b] \to \mathbb{R}$ and $g : [a, b] \to \mathbb{R}$ are continuous (and therefore integrable) functions. Show that if $\int_a^b f = \int_a^b g$, then there exists $c \in [a, b]$ such that $f(c) = g(c)$.

(b) Let $u, v : [a, b] \to \mathbb{R}$ be two continuous functions. Assume that $v \geq 0$ on $[a, b]$. Show that there exists $y \in [a, b]$ such that

$$\int_a^b u(x)v(x)\mathrm{d}x = u(y)\int_a^b v(x)\mathrm{d}x.$$

Proof.

(a) Since f and g are continuous and integrable, then we have

$$\inf f \leq \frac{1}{b-a}\int_a^b f \leq \sup f,$$

i.e.

$$\min f \leq \frac{1}{b-a}\int_a^b f \leq \max f.$$

By Intermediate Value Theorem, if u is a number between $f(a)$ and $f(b)$, that is

$$\min(f(a), f(b)) < u < \max(f(a), f(b)),$$

then there exists a $c \in (a, b)$, such that $f(c) = u$. In our context, there exists $c \in [a.b]$, such that

$$f(c) = \frac{1}{b-a}\int_a^b f.$$

Similarly,

$$g(c) = \frac{1}{b-a}\int_a^b g.$$

Since $\int_a^b f = \int_a^b g$, then $f(c) = g(c)$.

(b) Since u is continuous on a closed bounded set, then we have

$$\min u \leq u(x) \leq \max u,$$

and hence

$$v(x)\min u \leq u(x)v(x) \leq v(x)\max u,$$

which implies

$$\int_a^b mv(x) \leq \int_a^b u(x)v(x) \leq \int_a^b Mv(x),$$

where $m = \min u$ and $M = \max u$.

If $v = 0$, this is trivial and nothing to prove. Otherwise if $v > 0$, then $\int_a^b v(x)\mathrm{d}x > 0$. Hence,

$$m \leq \frac{\int_a^b u(x)v(x)}{\int_a^b v(x)\mathrm{d}x} \leq M.$$

Since u is continuous, it attains all values between m and M, and therefore, it exists a $y \in [a, b]$ such that

$$u(y) = \frac{\int_a^b u(x)v(x)}{\int_a^b v(x)\mathrm{d}x},$$

and hence

$$\int_a^b u(x)v(x)\mathrm{d}x = u(y)\int_a^b v(x)\mathrm{d}x.$$

□

Question 3.

Let $f, g : [a, b] \to \mathbb{R}$ be two functions that satisfy

$$|f(x) - f(y)| \leq |g(x) - g(y)| \qquad \forall x, y \in [a, b].$$

Prove that

$$\sup_{[a,b]} f - \inf_{[a,b]} f \leq \sup_{[a,b]} g - \inf_{[a,b]} g.$$

Use this result to complete the proof of Theorem 2.22 in the notes, more precisely, show that $f : [a, b] \to \mathbb{R}$ is integrable then so is $|f|$.

Proof.

First note that

$$f(x) - f(y) \leq |g(x) - g(y)| \leq \max\{g(x), g(y)\} - \min\{g(x), g(y)\} \leq \sup_{[a,b]} g - \inf_{[a,b]} g.$$

Then this means

$$\sup_{[a,b]}(f(x) - f(y)) \leq f(x) - f(y) \leq \sup_{[a,b]} g - \inf_{[a,b]} g.$$

We know that

$$\sup_{[a,b]}(f(x) - f(y)) = \sup_{[a,b]} f(x) - \inf_{[a,b]} f(y),$$

and hence

$$\sup_{[a,b]} f - \inf_{[a,b]} f \leq \sup_{[a,b]} g - \inf_{[a,b]} g.$$

Note that the reverse triangle inequality states that

$$||f(x)| - |f(y)|| \leq |f(x) - f(y)|.$$

By the previous part, we have

$$\sup_{[a,b]} |f| - \inf_{[a,b]} |f| \leq \sup_{[a,b]} f - \inf_{[a,b]} f.$$

Hence,

$$\begin{aligned} U(|f|, P) - L(|f|, P) &= \sum_{k=1}^{n} (\sup |f| - \inf |f|) |I_k| \\ &\leq \sum_{k=1}^{n} (\sup f - \inf f) |I_k| \\ &= U(f, P) - L(f, P) < \varepsilon. \end{aligned}$$

Now we conclude that if $f : [a, b] \to \mathbb{R}$ is integrable, then so is $|f|$. $\square$