MA244 Analysis III Support Class - Week 2

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1 Week 2

Question 1.

Recall that given a bounded interval $I = [a, b]$, a partition Q is a refinement of P if every interval in the partition P can be written as a union of intervals in the partition Q. (As a convention we take all intervals in a partition to be closed and non-trivial, i.e., $[s, s]$ is not allowed. The intervals are almost disjoint.)

(a) Given two partitions of $[a, b]$, say P and Q, assume that for every interval I_k in the partition P there exists at least one interval (say J_k) in the partition Q such that $J_k \subset I_k$. Is Q a refinement of P?

(b) Show that Q is a refinement of P if and only if for every interval Q_k in the partition Q there exists P_j in P such that $Q_k \subset P_j$.

Solution.

(a) Not necessarily true. Consider $P = \{[0, 1], [1, 2]\}$ and $Q = \left\{\left[0, \frac{1}{2}\right], \left[\frac{1}{2}, 1\right], \left[1, \frac{3}{2}\right], \left[\frac{3}{2}, 2\right]\right\}$. For each interval J_k, we have

$$\begin{cases} J_1, J_2 \subset I_1 \\ J_3, J_4 \subset I_2 \end{cases},$$

while $\left[\frac{1}{2}, \frac{3}{2}\right] \not\subseteq P$, meaning Q is not a refinement of P.

(b) If Q is a refinement of P, then there exists some j such that $Q_k \cap P_j$ is an interval of positive length. Also by definition, $P_j = \bigcup_{i=\alpha_j}^{\beta_j} Q_i$ for some $\alpha_j \le \beta_j$, and this implies that $Q_k = Q_i \subset P_j$.

If every interval Q_k in the partition Q, there exists $P_j \subset P$ such that $Q_k \subset P_j$, then we claim that $\bigcup_{k \in \mathbb{R}} Q_k = P_j$, and we want to prove it is true. Assume that it is not true. Then we have a closed set $S = \bigcup_{k: Q_k \subset P_j} Q_k$ such that $S \subset P_j$, but $P_j \setminus S$ has a positive length. By definition of a partition, there is $Q_i \not\subseteq P_j : Q_i \cap P_j$ is non-trivial and this means that $Q_i \not\subseteq P_l$ for any P_l in P. That is a contradiction and we conclude that $S = P_j$.

□

Question 2.

The oscillation of a bounded function f on a set A is defined by

$$\operatorname*{osc}_A f = \sup_A f - \inf_A f.$$

Let $f, g : [a, b] \to \mathbb{R}$ be bounded functions, with g Riemann integrable on $[a, b]$. Show that if there exists $C > 0$ such that

$$\operatorname*{osc}_I f \leq C \cdot \operatorname*{osc}_I g$$

on every interval $I \subset [a, b]$, then f is Riemann integrable.

Proof.

Since g is Riemann integrable on $[a, b]$, then for every $\varepsilon > 0$, there exists a partition P of $[a, b]$ such that

$$U(g, P) - L(g, P) < \varepsilon.$$

By definition,

$$\begin{aligned}
U(f, P) - L(f, P) &= \sum_{k=1}^{n} M_k |I_k| - \sum_{k=1}^{n} m_k |I_k| \\
&= \sum_{k=1}^{n} \sup_{I_k} f |I_k| - \sum_{k=1}^{n} \inf_{I_k} f |I_k| \\
&= \sum_{k=1}^{n} (\sup_{I_k} f - \inf_{I_k} f) |I_k| \\
&= \sum_{k=1}^{n} \operatorname*{osc}_{I_k} f |I_k| \\
&\leq \sum_{k=1}^{n} C \cdot \operatorname*{osc}_{I_k} g |I_k| \\
&= C \sum_{k=1}^{n} (\sup_{I_k} g - \inf_{I_k} g) |I_k| \\
&= C \left(U(g, P) - L(g, P) \right) \\
&< C\varepsilon < \varepsilon.
\end{aligned}$$

Therefore, f is Riemann integrable. □

Question 3.

Let $f : [a, b] \to \mathbb{R}$ be an integrable function. Prove that given $\varepsilon > 0$, there exists a partition of $[a, b]$, say $P = \{a = x_0 < x_1 < ... < x_{n-1} < x_n = b\}$ for some $n \in \mathbb{N}$, such that for any choice of points $c_k \in [x_{k-1}, x_k]$, $k = 1, 2, ..., n$ we have

$$\left| \int_a^b f(x) \mathrm{d}x - \sum_{k=1}^{n} f(c_k) \left[x_k - x_{k-1} \right] \right| < \varepsilon.$$

Proof.

By definition, if $f : [a, b] \to \mathbb{R}$ is integrable, then there exists a partition P such that

$$U(f, P) - L(f, P) < \varepsilon.$$

Also recall that there is a relation:

$$L(f, P) \leq \int_a^b f(x) \mathrm{d}x \leq U(f, P).$$

Hence,

$$0 \leq \int_a^b f(x)\mathrm{d}x - L(f,P) \leq U(f,P) - L(f,P) < \varepsilon.$$

Recall that

$$\begin{aligned} L(f,P) = \sum_{k=1}^{n} m_k |I_k| &= \sum_{k=1}^{n} \inf_{I_k} f \left[x_k - x_{k-1}\right] \\ &\leq \sum_{k=1}^{n} f(c_k) \left[x_k - x_{k-1}\right]. \end{aligned}$$

Therefore,

$$\left| \int_a^b f(x)\mathrm{d}x - \sum_{k=1}^{n} f(c_k) \left[x_k - x_{k-1}\right] \right| \leq \left| \int_a^b f(x)\mathrm{d}x - L(f,P) \right| < \varepsilon.$$

□