MA244 - Analysis III

October 5, 2022

Contents

About the notes

Analysis III is a self-contained module. Only material lectured in class will be examined. These notes are based on the lectured material and any content that is not examinable will be clearly marked.

The course is essentially divided into three parts, covering the following topics:

- Riemann integration,
- sequences and series of function, and
- complex valued functions.

The notes have been developed and adapted to the previous analysis modules from Year 2, using multiple sources. I do not know of a single source that covers any of the topics as presented here. I would be happy to supply a list of references that can be used to expand any of the Chapters in the notes.

Chapter 1

Riemann Integration

In this Chapter we are going to carefully define the notion of Riemann integral. Loosely speaking, given a function $f : [a, b] \to \mathbb{R}$ we think of the Riemann integral as the (signed) area underneath the graph.

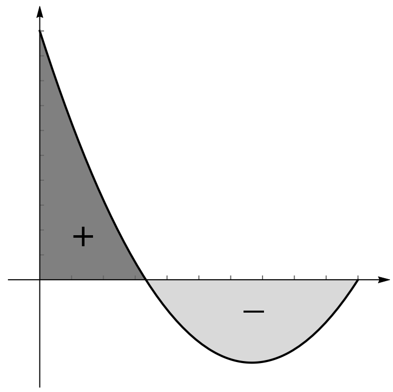

Figure 1.1: Integral as signed area under the curve

By *signed area* we mean that in fact we will allow negative values. For a function that changes sign like the one in Figure 1.1, the region in dark grey will have positive integral (area), while the region in light grey will have negative integral (area).

Before a formal definition, we will consider the function $f(x) = x^2$ and calculate the area under its graph, between 0 and 1. We will follow the approach of the ancient Greeks of computing an area by the exhaustion method.

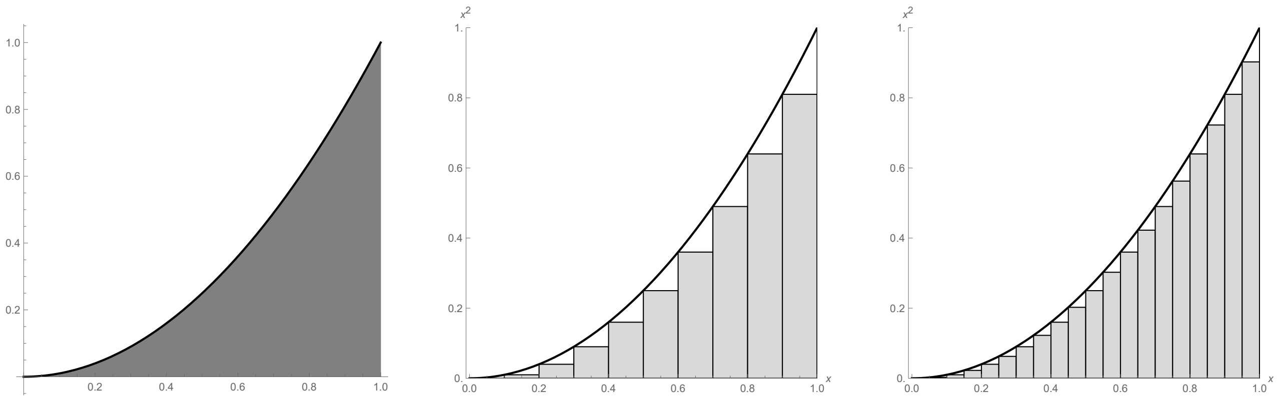

Figure 1.2: Calculating $\int_0^1 x^2 \mathrm{d}x$ by exhaustion

For this purpose we approximate the region (dark grey in Figure 1.2) by rectangles. In the Figure the interval $[0, 1]$ has been decomposed in 10 (centre) and 20 (right) intervals and the largest possible rectangles (with sides parallel to the axis) with those intervals as base have been drawn.

In general, let's assume that we decompose $[0,1]$ into k intervals of equal length, generating the family of intervals $[j/k,(j+1)/k]$, for $j=0,\ldots,k-1$. For the interval $[j/k,(j+1)/k]$ the height of the corresponding rectangle would be $(j/k)^2$, as the function x^2 is increasing in $[0,1]$. Therefore the area of the family of rectangles becomes

$$\sum_{j=0}^{k-1} \frac{1}{k}\frac{j^2}{k^2} = \frac{1}{k^3}\sum_{j=0}^{k-1} j^2 = \frac{1}{k^3}\frac{(k-1)k(2k-1)}{6},$$

where we have used that $\displaystyle\sum_{j=0}^{k} j^2 = \frac{k(k+1)(2k+1)}{6}$.

By considering partitions with an increasing number of intervals it *seems* clear that we approximate the desired area more and more closely. Therefore by taking limits as k tends to infinity we should obtain the right area. Indeed,

$$\lim_{k\to\infty}\frac{1}{k^3}\frac{(k-1)k(2k-1)}{6} = \frac{1}{3},$$

which is what we would expect to obtain using anti-derivatives to compute integrals, i.e.,

$$\int_0^1 x^2 \mathrm{d}x = \frac{1}{3}x^3\Big|_0^1 = \frac{1}{3}.$$

The approach of integration as anti-differentiation which you might have seen earlier runs into trouble easily when considering integrals like

$$\int_1^2 \mathrm{e}^{-x^2}\mathrm{d}x, \qquad \int_0^t \frac{1}{\sqrt{1-k^2\sin\theta}}\,\mathrm{d}\theta, \qquad \int_0^{2\pi}\sqrt{a^2\sin^2 t + b^2\cos^2 t}\,\mathrm{d}t$$

for $0<k^2<1$ and $a^2\neq b^2$. The last two integrals above are examples of elliptic integrals. Another important example is given by the Gamma function, $\Gamma(z)=\int_0^\infty x^{z-1}\mathrm{e}^{-x}\mathrm{d}x$.

1.1 Definition of the Riemann integral

We begin with the notion of *partition* of an interval. For convenience we will consider closed intervals. We say that two intervals are almost-disjoint if they have at most one common point. We call an interval $[c,d]$ non-trivial, if $c<d$.

Definition 1.1. *Let I be a non-trivial, closed interval in $\mathbb{R}$. A partition of I is a collection $\{I_1,\ldots,I_n\}$ of almost-disjoint, non-trivial closed intervals whose union is I.*

In practice, a partition of the interval $[a,b]$ is determined by a collection of points $\{x_i\}_{i=0}^n$, for some n such that

$$a=x_0<x_1<\ldots<x_{n-1}<x_n=b,$$

which correspond to the collection of intervals $I_j=[x_{j-1},x_j]$, for $j=1,\ldots,n$.

We will restrict ourselves to bounded functions, say $f:[a,b]\to\mathbb{R}$. Notice that several of the objects we define below will automatically be infinite if the functions are unbounded. Given a partition of $P=\{I_1,\ldots,I_n\}$ of $I=[a,b]$ we denote

$$M=\sup_I f \qquad m=\inf_I f \qquad M_k=\sup_{I_k} f \qquad m_k=\inf_{I_k} f.$$

Definition 1.2. *Given $f : [a,b] \to \mathbb{R}$ and a partition $P = \{I_1, \dots, I_n\}$ of $[a,b]$ we define the upper Riemann sum of f with respect to P as*

$$U(f,P) := \sum_{k=1}^{n} M_k |I_k|,$$

and the lower Riemann sum of f with respect to P as

$$L(f,P) := \sum_{k=1}^{n} m_k |I_k|.$$

Notice that $\sum_{k=1}^{n} |I_k| = |I| = b-a$ and $m \le m_k \le M_k \le M$ for $1 \le k \le n$, which leads to

$$m(b-a) = \sum_{k=1}^{n} m|I_k| \le \sum_{k=1}^{n} m_k|I_k| \le \sum_{k=1}^{n} M_k|I_k| \le \sum_{k=1}^{n} M|I_k| = M(b-a)$$

from which we deduce

$$m(b-a) \le L(f,P) \le U(f,P) \le M(b-a).$$

For the example we considered before, $f(x) = x^2$, Figure 1.3 shows the rectangles used in calculating the Lower and Upper Riemann sums, for a uniform partition with 10 intervals.

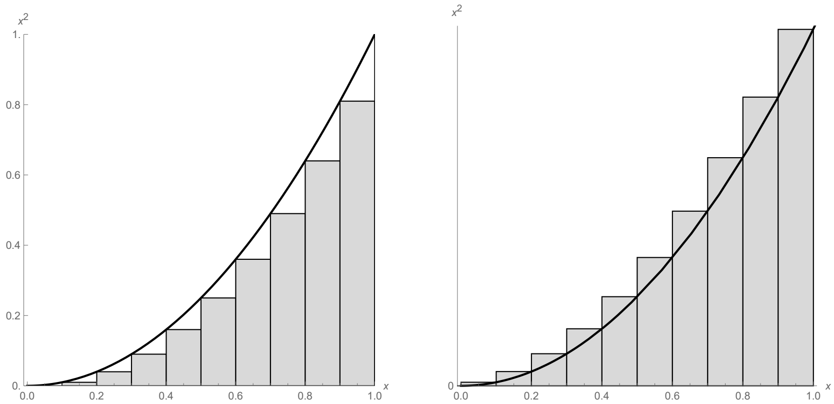

Figure 1.3: Lower (left) and Upper (right) Riemann sum of f

We will denote by $\mathscr{P}$ the set of all partitions of $[a,b]$.

Definition 1.3. *Given $f : [a,b] \to \mathbb{R}$, bounded, we define the upper Riemann integral of f by*

$$U(f) := \inf_{P \in \mathscr{P}} U(f,P).$$

We define the lower Riemann integral of f by

$$L(f) := \sup_{P \in \mathscr{P}} L(f,P).$$

Intuitively we see that $U(f,P)$ should be bigger than the area we are trying to calculate (if it exists), and we take an $\inf$ in the definition to find "the smallest" possible value. Similarly, $L(f,P)$ should be smaller than the area and taking a $\sup$ we aim to find "the largest" of those values.

Definition 1.4. *Given $f : [a,b] \to \mathbb{R}$ bounded we say that it is Riemann integrable if and only if $L(f) = U(f)$, and define its Riemann integral, denoted by $\int_a^b f(x)\mathrm{d}x$ or $\int_a^b f$, by*

$$\int_a^b f(x)\mathrm{d}x := L(f) = U(f).$$

We note that unbounded functions are not Riemann integrable. Notice that if we assume that f is, say, unbounded above, for any partition P we always have $U(f,P) = \infty$. Later on we will extend the definition to allow some unbounded functions by using a limiting procedure.

Exercise 1.1. *Calculate the integral of the following functions between 0 and 1, if it exists.*

$$f(x) = 1, \qquad g(x) = \begin{cases} 1 & 0 < x \leq 1 \\ 0 & x = 0 \end{cases}, \qquad \chi_{\mathbb{Q}}(x) := \begin{cases} 1 & x \in \mathbb{Q} \\ 0 & x \notin \mathbb{Q} \end{cases}.$$

Using the definition to calculate Riemann integrals is in practice very difficult, as one has to consider all partitions of $[a,b]$. Notice that in fact, given two different partitions we do not know (yet) that

$$L(f,P) \leq U(f,Q).$$

One would intuitively expect this to be the case, as when f is Riemann integrable we would expect

$$L(f,P) \leq \int_a^b f \leq U(f,Q).$$

To address this, we start by considering the notion of refinement.

Definition 1.5. *A partition $Q = \{J_1, \ldots, J_l\}$ of $[a,b]$ is a refinement of a partition $P = \{I_1, \ldots, I_n\}$ if every interval I_k in P is the union of one or more intervals J_k from the partition Q.*

If we think of the partitions P and Q in terms of the endpoints of the intervals that define them, Q is a refinement of P if all the endpoints defining P are in the collection of endpoints defining Q. Notice that given any two partitions of an interval it is possible that neither one is a refinement of the other.

Theorem 1.6. *Let $f : [a,b] \to \mathbb{R}$ be a bounded function and P and Q partitions of $[a,b]$, with Q a refinement of P. Then*

$$L(f,P) \leq L(f,Q) \leq U(f,Q) \leq U(f,P).$$

This can be understood *as: when we refine the partition we get better approximations*, which should converge if f is Riemann integrable.

Proof. Assume that $P = \{I_1, \ldots, I_n\}$ and $Q = \{J_1, \ldots, J_l\}$. We denote by m_i and M_i the infimum and the supremum of f in I_i and by $\bar{m}_j$ and $\bar{M}_j$ the infimum and the supremum of f in J_j.

Since we know that Q is a refinement of P we know that every interval I_i can be written as the union of almost-disjoint intervals J_j. In particular we can write, for some natural numbers α_i and β_i with $1 \leq \alpha_i \leq \beta_i \leq l$

$$I_i = \bigcup_{j=\alpha_i}^{\beta_i} J_j,$$

with $|I_i| = \sum_{j=\alpha_i}^{\beta_i} |J_j|$. Notice that this implies the relationships

$$m_i \leq \bar{m}_j \qquad \bar{M}_j \leq M_i \qquad \text{for } \alpha_i \leq j \leq \beta_i.$$

Therefore

$$L(f,P) = \sum_{i=1}^{n} m_i |I_i| = \sum_{i=1}^{n} m_i \sum_{j=\alpha_i}^{\beta_i} |J_j| \leq \sum_{i=1}^{n} \sum_{j=\alpha_i}^{\beta_i} \bar{m}_j |J_j| = \sum_{1}^{l} \bar{m}_j |J_j| = L(f,Q).$$

Similarly

$$U(f,P) = \sum_{i=1}^{n} M_i |I_i| = \sum_{i=1}^{n} M_i \sum_{j=\alpha_i}^{\beta_i} |J_j| \geq \sum_{i=1}^{n} \sum_{j=\alpha_i}^{\beta_i} \bar{M}_j |J_j| = \sum_{1}^{l} \bar{M}_j |J_j| = U(f,Q).$$

Since we already know that $L(f,Q) \leq U(f,Q)$ combining the above two inequalities completes the proof. □

As a consequence we can obtain the following result:

Theorem 1.7. *Let* $f : [a, b] \to \mathbb{R}$ *be a bounded function and* P, Q *two partitions of* $[a, b]$*. Then*

$$L(f, P) \leq U(f, Q)$$

Proof. We set R to be a refinement of P and Q. This can be easily achieved by defining the partition given by the union of all endpoints defining P and Q. By Theorem 1.6, applied first to P and R for the Lower Riemann sum and then to Q and R for the Upper Riemann sum, we have

$$L(f, P) \leq L(f, R) \leq U(f, R) \leq U(f, Q),$$

which is the desired result. □

A trivial Corollary from the above result is

Corollary 1.8. *Given* $f : [a, b] \to \mathbb{R}$ *bounded we have*

$$L(f) \leq U(f).$$

Exercise 1.2. *Prove the Corollary above. The proof only relies on basic properties of* $\sup$ *and* $\inf$*.*

The following result will also prove useful in showing that a function is integrable.

Theorem 1.9. *Let* $f : [a, b] \to \mathbb{R}$ *be a bounded function. Then* f *is integrable if and only if for every* $\varepsilon > 0$ *there exists a partition* P *of* $[a, b]$ *such that*

$$U(f, P) - L(f, P) < \varepsilon$$

Proof. By the properties of $\sup$ and $\inf$ in the respective definitions we know that there exists partitions P_1 and P_2 such that

$$U(f, P_1) < U(f) + \frac{\varepsilon}{2} \qquad L(f, P_2) > L(f) - \frac{\varepsilon}{2}.$$

Therefore if we consider the partition P which is a refinement of both P_1 and P_2, (for example by considering the union of all the endpoints of both partitions) we have

$$U(f, P) \leq U(f, P_1) < U(f) + \frac{\varepsilon}{2} \qquad L(f, P) \geq L(f, P_2) > L(f) - \frac{\varepsilon}{2}.$$

Notice that if f is integrable we have $U(f) = L(f)$, which implies, using the inequality above that

$$U(f, P) - L(f, P) < \varepsilon.$$

For the other implication notice that since

$$U(f) - L(f) \leq U(f, P) - L(f, P)$$

for every partition P the left-hand side, which is always greater or equal to zero, must be zero, as we can find partitions P that make it arbitrarily small. □

In fact, it is possible to give a *sequential* characterisation of the Riemann integral, which we leave as an Exercise.

Theorem 1.10. *Let* $f : [a, b] \to \mathbb{R}$ *be a bounded function.* f *is integrable if and only if there exists a sequence of partitions* P_n *such that*

$$\lim_{n\to\infty} U(f, P_n) - L(f, P_n) = 0.$$

We will now prove that continuous functions are Riemann integrable.

Theorem 1.11. *Let $f : [a, b] \to \mathbb{R}$ be a continuous function. Then it is Riemann integrable.*

Before we present a proof we review the notion of continuity and introduce *uniform continuity*.

Definition 1.12. *Given $f : [a, b] \to \mathbb{R}$, we say that f is continuous at $x \in [a, b]$ if for every $\varepsilon > 0$ there exists $\delta = \delta(x, \varepsilon) > 0$ such that*

$$y \in [a, b] \text{ and } |x - y| < \delta \Longrightarrow |f(y) - f(x)| < \varepsilon. \tag{1.1}$$

For the endpoints a and b we can only talk about one-sided continuity. We will say that $f : [a, b] \to \mathbb{R}$ is continuous on $[a, b]$ if it is continuous for every $x \in [a, b]$, with the endpoints understood as one-sided continuity. The key point to note from the definition above is that given a function f, $\varepsilon > 0$ and a point x there exists δ, but δ can depend on ε and x (and of course f).

Definition 1.13. *Given $f : [a, b] \to \mathbb{R}$, we say that it is uniformly continuous if for every $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon) > 0$ such that*

$$x, y \in [a, b] \text{ and } |x - y| < \delta \Longrightarrow |f(y) - f(x)| < \varepsilon. \tag{1.2}$$

The key point here is that δ can be chosen independently of x. In proving Theorem 1.11 we will also need the following result concerning continuous functions on a closed bounded interval.

Theorem 1.14. *Let $f : [a, b] \to \mathbb{R}$ be a continuous function. Then it is uniformly continuous.*

Before we prove the result let's consider a couple of examples in which the closed, bounded interval $[a, b]$ is replaced by an unbounded or an open domain.

Consider $f(x) = \mathrm{e}^x$, defined in $\mathbb{R}$. Clearly this is a continuous function, but not uniformly continuous. Indeed, since f grows faster and faster for larger x it is possible to find arbitrarily small intervals in which f changes by at least ε. This example shows that the result in Theorem 1.14 is not necessarily true for unbounded domains.

We can also consider $g(x) = \frac{1}{x}$ on $(0, 1)$. Just as in the previous example, near zero, the function g grows to infinity faster and faster as we approach the origin, making it impossible to find δ independent of x that satisfies (1.2).

This result, in much more generality, not just for closed intervals on $\mathbb{R}$ will be proven in MA260 Norms, Metrics and Topologies. The key point is that the domain of f is a *compact* set (which in this case is equivalent to close and bounded).

Proof of Theorem 1.14. We will argue by contradiction. That would mean that there exist $\varepsilon > 0$ and x_n, y_n such that $|x_n - y_n| \leq \frac{1}{n}$ but $|f(x_n) - f(y_n)| > \varepsilon$.

The sequences $\{x_n\}$ and $\{y_n\}$ are bounded, as they are in $[a, b]$, and therefore we can apply Bolzano–Weierstrass[1] to obtain convergent subsequences $\{x_{n_k}\}_{k=1}^{\infty}$ to x and $\{y_{n_k}\}_{k=1}^{\infty}$ to y. As $[a, b]$ is closed, $x, y \in [a, b]$.

Notice that

$$|x - y_{n_k}| \leq |x - x_{n_k}| + |x_{n_k} - y_{n_k}| \leq |x - x_{n_k}| + \frac{1}{n_k} \xrightarrow[k \to \infty]{} 0,$$

which implies that $x = y$. However we know that $|f(x_{n_k}) - f(y_{n_k})| > \varepsilon$ for all k. Since f is continuous at $x \in [a, b]$, taking limits as k goes to infinity we obtain $0 = |f(x) - f(x)| > \varepsilon$, which is a contradiction. □

We now turn to the Proof of Theorem 1.11.

[1] Theorem: Bolzano–Weierstrass. In $\mathbb{R}^n$ every bounded sequence has a convergent subsequence.

Proof of Theorem 1.11. Theorem 1.14 implies that f is in fact uniformly continuous. Therefore, given any $\varepsilon > 0$ there exists $\delta > 0$ such that

$$x, y \in [a, b] \text{ and } |x - y| < \delta \Longrightarrow |f(x) - f(y)| < \frac{\varepsilon}{b - a}.$$

We are going to prove that f is Riemann integrable using Theorem 1.9. Given any $\varepsilon > 0$ we need to find a partition of $[a, b]$ such that $U(f, P) - L(f, P) < \varepsilon$. We pick any partition $P = \{I_1, \ldots, I_n\}$ that satisfies $|I_k| < \delta$, where this δ comes from the uniform continuity of f as discussed above.

Now, on each interval I_k, since f is continuous, we know that the maximum and minimum of f exist and are attained. Say, with our earlier notation that

$$M_k = f(x_k), \qquad m_k = f(y_k) \qquad \text{with } x_k, y_k \in I_k$$

and as a consequence $M_k - m_k = f(x_k) - f(y_k) < \dfrac{\varepsilon}{b-a}$. Hence

$$U(f, P) - L(f, P) = \sum_{k=1}^{n} M_k |I_k| - \sum_{k=1}^{n} m_k |I_k| = \sum_{k=1}^{n} (M_k - m_k)|I_k| < \frac{\varepsilon}{b - a} \sum_{k=1}^{n} |I_k| = \varepsilon,$$

completing the proof. □

The class of Riemann integrable functions also includes monotonic (not necessarily continuous) functions.

Theorem 1.15. *Let $f : [a, b] \to \mathbb{R}$ be a monotonic function. Then it is Riemann integrable.*

Proof. We prove the result for monotone increasing functions, with the decreasing case being completely analogous. We will use Theorem 1.9 to prove the integrability of f. We consider a uniform partition of $[a, b]$ into n intervals, that we denote I_k. Now $I_k = [a + \frac{k}{n}(b-a), a + \frac{k+1}{n}(b-a)]$ for $k = 0, \ldots, n-1$. Since f is increasing, its infimum on I_k is bounded below by the value of f at the left endpoint, the supremum on I_k is bounded above by the value of f the right endpoint. Therefore,

$$U(f, P) - L(f, P) = \sum_{k=0}^{n-1} M_k |I_k| - \sum_{k=0}^{n-1} m_k |I_k| \le \frac{b - a}{n} \sum_{k=0}^{n-1} \left[f(a + \frac{k + 1}{n}(b - a)) - f(a + \frac{k}{n}(b - a)) \right]$$

but this last sum is just a telescopic sum that equals $f(b) - f(a)$, which yields

$$U(f, P) - L(f, P) \le \frac{b - a}{n}[f(b) - f(a)].$$

It is clear that given any ε we can choose n large enough so that $\frac{b-a}{n}[f(b) - f(a)] \le \varepsilon$, obtaining the result. □

This Theorem means that the we can construct a function with infinitely many (countably many) discontinuities that is integrable. Indeed, consider any enumeration of the rationals in $[0, 1]$, denoted by $\{r_k\}_1^\infty$. Define

$$f(x) := \sum_{k \text{ such that } r_k < x} \frac{1}{2^k},$$

with $f(0) = 0$. The function is clearly increasing, as the number of terms in the sum increases with x. Therefore it is integrable. The function does have a jump at every rational point in $[0, 1]$.

1.2 Fundamental properties of the Riemann integral

In this section we will concentrate on the fundamental properties of the Riemann function. We start by showing linearity, monotonicity and additivity.

Theorem 1.16. *Let $f, g : [a, b] \to \mathbb{R}$ be Riemann integrable functions, and $c \in \mathbb{R}$. Then $f + g$ and cf are Riemann integrable and we have*

$$\int_a^b cf = c\int_a^b f, \qquad \int_a^b (f+g) = \int_a^b f + \int_a^b g.$$

Proof. We start by considering $\int_a^b cf = c\int_a^b f$ for $c \geq 0$. Notice that in this case

$$\sup_I cf = c\sup_I f \qquad \inf_I cf = c\inf_I f,$$

from which it follows that for any partition P of $[a, b]$

$$U(cf, P) = cU(f, P) \qquad L(cf, P) = cL(f, P)$$

and therefore

$$U(cf) = \inf_{p\in\mathscr{P}} U(cf, P) = c\inf_{p\in\mathscr{P}} U(f, P) = cU(f)$$
$$L(cf) = \sup_{p\in\mathscr{P}} L(cf, P) = c\sup_{p\in\mathscr{P}} L(f, P) = cL(f).$$

Now, if f is integrable, it follows that $U(f) = L(f)$, and the above two equalities yields $U(cf) = L(cf) = cU(f) = cL(f)$, from which, $\int_a^b cf = c\int_a^b f$ follows.

Notice that for $c < 0$, suffices to prove the result for $c = -1$, and then apply the previous part. For the case $c = -1$ we have

$$\sup_I -f = -\inf_I f \qquad \inf_I -f = -\sup_I f.$$

Therefore

$$U(-f, P) = -L(f, P) \qquad L(-f, P) = -U(f, P)$$

and so

$$U(-f) = \inf_{p\in\mathscr{P}} U(-f, P) = \inf_{p\in\mathscr{P}} -L(f, P) = -\sup_{p\in\mathscr{P}} L(f, P) = -L(f),$$
$$L(-f) = \sup_{p\in\mathscr{P}} L(-f, P) = \sup_{p\in\mathscr{P}} -L(f, P) = -\inf_{p\in\mathscr{P}} U(f, P) = -U(f).$$

As before, if f is integrable $U(f) = L(f)$ from which we obtain $U(-f) = L(-f) = -U(f) = -L(f)$ and the result follows.

Now we turn our attention to $\int_a^b (f+g) = \int_a^b f + \int_a^b g$. Notice that for a partition $P = \{I_1, \ldots, I_n\}$ of $[a, b]$

$$U(f+g, P) = \sum_{k=1}^n \sup_{I_k}(f+g)|I_k| \leq \sum_{k=1}^n \sup_{I_k}(f)|I_k| + \sum_{k=1}^n \sup_{I_k}(g)|I_k| = U(f, P) + U(g, P).$$

Now since $U(f)$ is the infimum of $U(f, P)$ for all partitions, given any $\varepsilon > 0$ there exist partitions P_1 and P_2 such that

$$U(f, P_1) < U(f) + \frac{\varepsilon}{2} \qquad U(g, P_2) < U(g) + \frac{\varepsilon}{2}.$$

Consider a refinement of P_1 and P_2, and denote it by Q. We have

$$U(f+g) \leq U(f+g, Q) \leq U(f, Q) + U(g, Q) \leq U(f) + U(g) + \varepsilon$$

since $U(f,Q) \leq U(f,P_1)$ and $U(g,Q) \leq U(g,P_2)$. Since this holds for every $\varepsilon > 0$, we obtain $U(f+g) \leq U(f) + U(g)$.

Similarly

$$L(f+g,P) = \sum_{k=1}^{n} \inf_{I_k}(f+g)|I_k| \geq \sum_{k=1}^{n} \inf_{I_k}(f)|I_k| + \sum_{k=1}^{n} \inf_{I_k}(g)|I_k| = L(f,P) + L(g,P).$$

Now since $L(f)$ is the supremum of $L(f,P)$ for all partitions, given any $\varepsilon > 0$ there exist partitions P_1 and P_2 such that

$$L(f) - \frac{\varepsilon}{2} < L(f,P_1) \qquad\qquad L(g) - \frac{\varepsilon}{2} < L(f,P_2).$$

As before, considering a refinement of P_1 and P_2, denoted by Q, we have

$$L(f+g) \geq L(f+g,Q) \geq L(f,Q) + L(g,Q) > L(f) + L(g) - \varepsilon,$$

and taking limits as ε goes to zero we obtain $L(f+g) \geq L(f) + L(g)$. Therefore

$$U(f+g) \leq U(f) + U(g) = L(f) + L(g) \leq L(f+g).$$

Since $L(f+g) \leq U(f+g)$ all inequalities above are actually equalities and the result follows. □

Exercise 1.3. *We have shown above that for integrable functions $U(f+g) = U(f) + U(g)$ and similarly $L(f+g) = L(f)+L(g)$. This is however not necessarily the case when f and g are not Riemann integrable. Construct a pair of functions for which the equalities do not hold. Hint: you might consider the indicator of the rationals as one of the building blocks.*

Theorem 1.17. *Let $f, g : [a,b] \to \mathbb{R}$ be Riemann integrable functions such that $f \leq g$. Then*

$$\int_a^b f \leq \int_a^b g.$$

Proof. By the previous Theorem we know that $g - f$ is integrable and that $\int_a^b (g-f) = \int_a^b g - \int_a^b f$. Since $g - f \geq 0$ we have $U(g-f,P) \geq 0$, as the supremum of $g - f$ is greater or equal to zero. Therefore $U(g-f) \geq 0$, which means that

$$0 \leq \int_a^b (g-f) = \int_a^b g - \int_a^b f,$$

obtaining the desired result. □

Remark 1.18. *Notice that once we have established linearity, monotonicity has reduced to showing positivity. That is, showing that if $f \geq 0$ is Riemann integrable then $\int f \geq 0$.*

The following results are simple consequence of monotonicity.

Corollary 1.19. *Let $f : [a,b] \to \mathbb{R}$ be integrable. Let $m = \inf f$ and $M = \sup f$ (both on $[a,b]$). Then*

$$m(b-a) \leq \int_a^b f \leq M(b-a).$$

Corollary 1.20. *Let $f : [a,b] \to \mathbb{R}$ be a continuous function. Then there exists $c \in [a,b]$ such that*

$$f(c) = \frac{1}{b-a} \int_a^b f.$$

Proof. Notice that with the notation of Corollary 1.19 we have

$$m \leq \frac{1}{b-a}\int_a^b f \leq M.$$

Since f is continuous, the intermediate value theorem tells us that it attains every value in between m and M, and in particular $\frac{1}{b-a}\int_a^b f$. ☐

Remark 1.21. *The quantity $\frac{1}{b-a}\int_a^b f$ corresponds to the average of f on that interval $[a,b]$, or in probabilistic terms its expectation. If we wanted to replace f by a constant on that interval, in such a way that both functions had the same integral, $\frac{1}{b-a}\int_a^b f$ is the correct value.*

A final consequence of monotonicity is given in the following result.

Theorem 1.22. *Let $f : [a,b] \to \mathbb{R}$ be an integrable function. Then $|f|$ is integrable and we have*

$$\left|\int_a^b f\right| \leq \int_a^b |f|.$$

Proof. To show that $|f|$ is integrable we can use $\sup|f| - \inf|f| \leq \sup f - \inf f$ to show that for every $\varepsilon > 0$ there exists a partition P such that $U(|f|,P) - L(|f|,P) < \varepsilon$. The details are left as an exercise. The inequality follows from monotonicity of the integral noting that $-|f| \leq f \leq |f|$. ☐

Theorem 1.23. *Let $f : [a,b] \to \mathbb{R}$ and $c \in (a,b)$. Then f is Riemann integrable on $[a,b]$ if and only if it is Riemann integrable on $[a,c]$ and $[c,b]$. Moreover,*

$$\int_a^c f + \int_c^b f = \int_a^b f.$$

Proof. If f is integrable in $[a,b]$, for every $\varepsilon > 0$ there exists a partition P of $[a,b]$ such that

$$U(f,P) - L(f,P) < \varepsilon.$$

Set P_c to be the partition obtained by adding c to the partition P. (If c is already the endpoint of an interval in P, then $P_c = P$.) Let Q be the partition of $[a,c]$ induced by P_c, and R the partition of $[c,b]$ induced by P_c. Notice that we have

$$U(f,P_c) = U(f,Q) + U(f,R) \qquad L(f,P_c) = L(f,Q) + L(f,R),$$

which we can combine to obtain

$$U(f,Q) - L(f,Q) = \underbrace{U(f,P_c) - L(f,P_c)}_{<\varepsilon} - \underbrace{(U(f,R) - L(f,R))}_{\geq 0} < \varepsilon.$$

The estimate below the first curly bracket uses that P_c is a refinement of P and Theorem 1.6. This shows that f is Riemann integrable in $[a,c]$. Similarly

$$U(f,R) - L(f,R) = \underbrace{U(f,P_c) - L(f,P_c)}_{<\varepsilon} - \underbrace{(U(f,Q) - L(f,Q))}_{\geq 0} < \varepsilon,$$

showing that f is integrable in $[c,b]$.

For the converse, if f is integrable in $[a,c]$ and $[c,b]$ we know that for every ε there are partitions Q of $[a,c]$ and R of $[c,b]$ such that

$$U(f,Q) - L(f,Q) < \frac{\varepsilon}{2} \qquad U(f,R) - L(f,R) < \frac{\varepsilon}{2}. \tag{1.3}$$

Set P to be the partition of $[a,b]$ obtained by combining Q and R. Notice that P includes c as one of the endpoints. Therefore, as before (for P_c) we have

$$U(f,P)-L(f,P)=U(f,Q)-L(f,Q)+U(f,R)-L(f,R)<\varepsilon,$$

with the last inequality following from (1.3). Therefore f is integrable on $[a,b]$.

Now,

$$\int_a^b f \le U(f,P)=U(f,Q)+U(f,R)\le L(f,Q)+L(f,R)+\varepsilon \le \int_a^c f+\int_c^b f+\varepsilon,$$

$$\int_a^b f \ge L(f,P)=L(f,Q)+L(f,R)\ge U(f,Q)+U(f,R)-\varepsilon \ge \int_a^c f+\int_c^b f-\varepsilon.$$

Since ε is arbitrary the result follows by taking ε to zero. □

Theorem 1.24. *Let $f:[a,b]\to\mathbb{R}$ be a Riemann integrable function and $\varphi:\mathbb{R}\to\mathbb{R}$ a continuous function. Then $\varphi\circ f$ is Riemann integrable.*

Proof. Since f is integrable, it is bounded, say $|f|\le M$ we only need to consider φ on $[-M,M]$. On that interval φ is actually uniformly continuous (Theorem 1.14), and bounded, say $|\varphi|\le K$. That means that given $\bar\varepsilon>0$ there exists δ such that

$$\forall x,y\in[-M,M],\ \ |x-y|<\delta \Longrightarrow |\varphi(x)-\varphi(y)|<\bar\varepsilon. \tag{1.4}$$

Also, since f is integrable, given $\eta>0$ (to be chosen later) there exists a partition Q_η such that

$$U(f,Q_\eta)-L(f,Q_\eta)=\sum(\sup_{I_k} f-\inf_{I_k} f)|I_k|<\eta. \tag{1.5}$$

We want to show that for every $\varepsilon>0$ there exists a partition P such that

$$U(\varphi\circ f,P)-L(\varphi\circ f,P)<\varepsilon. \tag{1.6}$$

We will take P as one of the Q_η for an η to be chosen later.

$$U(\varphi\circ f,P)-L(\varphi\circ f,P)=\sum_{k=1}^{n}(\sup_{I_k}\varphi\circ f-\inf_{I_k}\varphi\circ f)|I_k|$$

$$=\sum_{\substack{k\\ \sup_{I_k} f-\inf_{I_k} f<\delta}}\underbrace{(\sup_{I_k}\varphi\circ f-\inf_{I_k}\varphi\circ f)}_{\le\bar\varepsilon \text{ by } (1.4)}|I_k|+\sum_{\substack{k\\ \sup_{I_k} f-\inf_{I_k} f\ge\delta}}\underbrace{(\sup_{I_k}\varphi\circ f-\inf_{I_k}\varphi\circ f)}_{|\varphi|\le K \text{ on } [-M,M]}|I_k|$$

$$\le\sum_{k=1}^{n}\bar\varepsilon|I_k|+\sum_{\substack{k\\ \sup_{I_k} f-\inf_{I_k} f\ge\delta}}2K|I_k|$$

$$\le\bar\varepsilon(b-a)+2K\frac{\eta}{\delta}.$$

In the last inequality we have used that

$$\sum_{\substack{k\\ \sup_{I_k} f-\inf_{I_k} f\ge\delta}}|I_k|\le\frac{\eta}{\delta}.$$

To see this, notice that since we have chosen Q_η satisfying (1.5)

$$\sum_{\substack{k \\ \sup_{I_k} f - \inf_{I_k} f \geq \delta}} \delta|I_k| \leq \sum_{\substack{k \\ \sup_{I_k} f - \inf_{I_k} f \geq \delta}} (\sup_{I_k} f - \inf_{I_k} f)|I_k| \leq \sum_k (\sup_{I_k} f - \inf_{I_k} f)|I_k| \leq \eta.$$

To complete the proof we choose all the parameters in the following order. First, given $\varepsilon > 0$ set

$$\bar{\varepsilon} = \frac{\varepsilon}{2(b-a)}.$$

The uniform continuity of φ, see (1.4), yields the corresponding δ to that $\bar{\varepsilon}$, and then we choose

$$\eta = \frac{\delta}{4K}\varepsilon,$$

from which we obtain the partition $P = Q_\eta$ that satisfies (1.6). □

Remark 1.25. *The composition of two Riemann integrable functions is NOT necessarily integrable. An example of a pair of integrable functions that show this is*

$$f(x) = \begin{cases} 1 & x \neq 0 \\ 0 & x = 0 \end{cases} \qquad g(x) = \begin{cases} \frac{1}{q} & x = \frac{p}{q} \text{ for } p, q \text{ coprime}, q > 0, \\ 1 & x = 0, \\ 0 & x \notin \mathbb{Q}. \end{cases}$$

The composition

$$f \circ g = \chi_{\mathbb{Q}} = \begin{cases} 1 & x \in \mathbb{Q} \\ 0 & x \notin \mathbb{Q} \end{cases},$$

which is not integrable. Proving that f is Riemann integrable is easy from the definition, while showing that g is Riemann integrable is bit harder, see the example sheet for Week 2.

The following statement is a direct consequence of Theorem 1.24:

Theorem 1.26. *Let $f, g : [a,b] \to \mathbb{R}$ be Riemann integrable functions. Then the product fg is Riemann integrable. If in addition $\frac{1}{g}$ is bounded then $\frac{f}{g}$ is Riemann integrable.*

Proof. We start by proving that f^2 is integrable. This follows directly from Theorem 1.24, choosing $\varphi(x) = x^2$. In order to consider fg we write

$$fg = \frac{1}{2}[(f+g)^2 - f^2 - g^2].$$

Every function on the right-hand side is integrable, as it is the square of an integrable function and, by linearity, so is fg. In order to show that $\frac{1}{g}$ is integrable, we notice that if it is bounded then g must be bounded away from zero, that is, there exists $\varepsilon > 0$ such that $\varepsilon < |g|$ on $[a,b]$. Now, consider the function φ

$$\varphi(x) = \begin{cases} \dfrac{1}{x} & |x| > \varepsilon, \\ \dfrac{1}{\varepsilon^2} x & |x| \leq \varepsilon. \end{cases}$$

Notice that $\varphi \circ g = \frac{1}{g}$ on the domain of g, namely on $[a,b]$. Since g is integrable, and φ is clearly continuous the integrability of $\varphi \circ g$ follows from Theorem 1.24. □

Remark 1.27. *Theorem 1.24 has been stated with φ defined on the whole of $\mathbb{R}$ for simplicity, but that is of course not necessary, as it only needs to be defined on $f([a,b])$, the image of $[a,b]$ by f. With that in mind, it would not have been necessary to extend the function φ linearly for $|x| \leq \varepsilon$ in the proof above.*

1.3 The Fundamental Theorem of Calculus

In an informal way, in this section we will study the relationship between integration and differentiation, and how under sufficient conditions they can be understood as inverse operations. The first result we consider is when the integral of a derivative is the original function.

Theorem 1.28. *Let $F : [a,b] \to \mathbb{R}$ be a continuous function that is differentiable on (a,b) with $F' = f$. Assume that $f : [a,b] \to \mathbb{R}$ is an integrable function. Then*

$$\int_a^b f(x)\mathrm{d}x = F(b) - F(a).$$

Proof. Notice that it suffices to show that

$$L(f,P) \leq F(b) - F(a) \leq U(f,P) \tag{1.7}$$

for every partition P of $[a,b]$. Indeed, by taking appropriate $\sup$ (for the left inequality) and $\inf$ (for the right inequality) we obtain $L(f) \leq F(b) - F(a) \leq U(f)$, but since f is integrable $L(f) = U(f)$ and therefore equal to $F(b) - F(a)$.

In order to prove (1.7), consider any partition P of the interval $[a,b]$ in terms of its endpoints, say $P = \{a = x_0, x_1, \dots, x_{n-1}, x_n = b\}$. Now on every interval $I_k := [x_{k-1}, x_k]$, for $k = 1, \cdots, n$ we have

$$\inf_{I_k} f(x)(x_k - x_{k-1}) \leq f(c_k)(x_k - x_{k-1}) \leq \sup_{I_k} f(x)(x_k - x_{k-1}) \tag{1.8}$$

for every $c_k \in (x_{k-1}, x_k)$. Since F is continuous on $[x_{k-1}, x_k]$ and differentiable on (x_{k-1}, x_k) by the Mean Value Theorem we know that there exists c_k such that $F(x_k) - F(x_{k-1}) = f(c_k)(x_k - x_{k-1})$. In particular that means that for the corresponding c_k, the inequality(1.8) becomes

$$\inf_{I_k} f(x)(x_k - x_{k-1}) \leq F(x_k) - F(x_{k-1}) \leq \sup_{I_k} f(x)(x_k - x_{k-1}). \tag{1.9}$$

Taking the sum in k from 1 to n, (1.9) yields

$$L(f,P) \leq \sum_{k=1}^{n} (F(x_k) - F(x_{k-1})) \leq U(f,P),$$

but since $\sum_{k=1}^{n} (F(x_k) - F(x_{k-1})) = F(b) - F(a)$ we obtain the desired result. □

Remark 1.29. *Notice that Theorem 1.28 only requires that F is differentiable on (a,b), not requiring that derivatives from that right (at a) or from the left (at b) exist. However since we require that F' is integrable, we do need that f is bounded. That means that the result does not apply for example to $F : [0,1] \to \mathbb{R}$, with $F(x) = \sqrt{x}$. This is despite the fact that you have most likely used*

$$\int_0^1 \frac{1}{2\sqrt{x}}\mathrm{d}x = \sqrt{x}\Big|_0^1 = \sqrt{1} - \sqrt{0} = 1$$

in previous years. We will address this issue when we consider improper integrals.

We now explore the opposite direction, by considering derivatives of integrals.

Theorem 1.30. *Let $f : [a,b] \to \mathbb{R}$ be an integrable function and define the function $F : [a,b] \to \mathbb{R}$ by*

$$F(x) := \int_a^x f(t)\mathrm{d}t.$$

Then F is continuous of $[a,b]$. Additionally if f is continuous at $c \in [a,b]$ then $F'(c) = f(c)$, with the derivatives at a and b understood as one-sided derivatives.

Proof. First, by the additivity Theorem 1.23 we know that f is integrable on the interval $[a, x]$ and therefore F is well defined. Also since f is integrable we know that it is bounded, say $|f| \leq M$, and therefore

$$F(x+h) - F(x) = \int_a^{x+h} f(t)\mathrm{d}t - \int_a^x f(t)\mathrm{d}t = \int_a^x f(t)\mathrm{d}t + \int_x^{x+h} f(t)\mathrm{d}t - \int_a^x f(t)\mathrm{d}t$$

$$= \int_x^{x+h} f(t)\mathrm{d}t,$$

from which $|F(x+h) - F(x)| \leq M|h|$, proving that F is (Lipschitz) continuous. Also, from that equality we deduce

$$\frac{F(x+h) - F(x)}{h} = \frac{1}{h}\int_x^{x+h} f(t)\mathrm{d}t.$$

The result will follow if we show that whenever f is continuous at x (in addition to integrable) we have

$$\lim_{h\to 0} \frac{1}{h}\int_x^{x+h} f(t)\mathrm{d}t = f(x).$$

Notice that since we are integrating with respect to t, $f(x)$ is a constant, and therefore

$$\frac{1}{h}\int_x^{x+h} f(t)\mathrm{d}t - f(x) = \frac{1}{h}\int_x^{x+h} (f(t) - f(x))\mathrm{d}t.$$

It suffices to show that with the hypotheses above on f

$$\lim_{h\to 0} \frac{1}{h}\int_x^{x+h} (f(t) - f(x))\mathrm{d}t = 0.$$

Now, since f is continuous at x, given any $\varepsilon > 0$, there exists $\delta > 0$ such that $|x - y| < \delta$ implies $|f(x) - f(y)| < \varepsilon$. Therefore, for $|h| < \delta$ we have

$$\lim_{h\to 0}\left|\frac{1}{h}\int_x^{x+h} (f(t) - f(x))\mathrm{d}t\right| \leq \lim_{h\to 0}\left|\frac{1}{h}\int_x^{x+h} |(f(t) - f(x))|\mathrm{d}t\right| \leq \lim_{h\to 0}\left|\frac{1}{h}\int_x^{x+h} \varepsilon\mathrm{d}t\right| = \varepsilon,$$

which implies, taking limits as ε tends to zero, the desired result. □

In the proof of the Theorem above we have proven the following result that we state here for future reference.

Theorem 1.31. *Let $f : [a, b] \to \mathbb{R}$ be an integrable function on $[a, b]$ and continuous (from the right) at a. Then*

$$\lim_{h\to 0^+} \frac{1}{h}\int_a^{a+h} f(t)\mathrm{d}t = f(a).$$

Similarly if f is continuous (from the left) at b we have

$$\lim_{h\to 0^+} \frac{1}{h}\int_{b-h}^{b} f(t)\mathrm{d}t = f(b).$$

In fact the result is more general and we can consider a family of intervals I_h, such that $x \in I_h$ and $|I_h| \to 0$ and have (assuming as above that f is continuous at x) that

$$\lim_{h\to 0} \frac{1}{|I_h|}\int_{I_h} f(t)\mathrm{d}t = f(x).$$

1.3.1 Consequences of the Fundamental Theorem of Calculus

One of the main consequences of the FTC is as a basic tool for computing integrals. Indeed, while there is no systematic procedure for doing this, to compute $\int f$ we want to find a function F such that $F' = f$, as justified by Theorem 1.28.

For example, since for $p > -1$ the derivative of $\frac{1}{p+1}x^{p+1}$ is x^p we obtain

$$\int_0^1 x^p \mathrm{d}x = \frac{1}{p+1}.$$

Note that so far we are only able to integrate bounded functions and therefore we only have the result for $p \geq 0$.

In this sense we can define the indefinite integral of a function f as the most general anti-derivative of f. If $F(x)$ is any anti-derivative of f, we would define the indefinite integral of f as

$$\int f(x)\mathrm{d}x = F(x) + c, \qquad c \text{ is any constant.}$$

c is usually referred to as the constant of integration. Note that using two different Fs that differ by a constant to calculate a definite integrals (i.e. with limits) using Theorem 1.28 would result in the same answer.

Another important consequence of the Fundamental Theorem of Calculus is the integration by parts formula.

Theorem 1.32. *Let $f, g : [a, b] \to \mathbb{R}$ be continuous functions on $[a, b]$ that are differentiable on (a, b), and such that f' and g' are integrable on $[a, b]$. Then*

$$\int_a^b f(x)g'(x)\mathrm{d}x = f(b)g(b) - f(a)g(a) - \int_a^b f'(x)g(x)\mathrm{d}x.$$

Proof. Notice that the derivative of fg exists and equals $f'g+fg'$, and that all the functions $f, g, fg, f'g, fg'$ and $(fg)'$ are integrable, either by assumption or because they are linear combinations of products of integrable functions. Therefore by Theorem 1.28 we obtain

$$\int_a^b (fg)' = f(b)g(b) - f(a)g(a),$$

and since $\int_a^b (fg)' = \int_a^b f'g + \int_a^b fg'$ we obtain the desired result. □

Another significant consequence of the Fundamental Theorem is the change of variable formula.

Theorem 1.33. *Let $f : [a, b] \to \mathbb{R}$ be a differentiable function (understood as one-sided on the end-points), such that f' is integrable on $[a, b]$. Let g be a continuous function on $f([a, b])$, the image of $[a, b]$ under the map f. Then*

$$\int_a^b g(f(x))f'(x)\mathrm{d}x = \int_{f(a)}^{f(b)} g(t)\mathrm{d}t. \tag{1.10}$$

Proof. Define, for $x \in f([a, b])$ the function $G(x) := \int_{f(a)}^x g(t)\mathrm{d}t$. Since g is continuous, Theorem 1.30 yields $G'(x) = g(x)$. Therefore, using the chain rule we obtain

$$G(f(x))' = g(f(x))f'(x).$$

We note that the right-hand side is integrable. Indeed, since g is continuous and f integrable, Theorem 1.24 ensures $g(f(x))$ is integrable; as f' is integrable, the right-hand side is the product of two integrable functions, and therefore integrable.

Now, using (1.10) and Theorem 1.28 we find

$$\int_a^b g(f(x))f'(x)\mathrm{d}x = \int_a^b G(f(x))'\mathrm{d}x = G(f(b))-G(f(a)) = \int_{f(a)}^{f(b)} g(t)\mathrm{d}t - \int_{f(a)}^{f(a)} g(t)\mathrm{d}t = \int_{f(a)}^{f(b)} g(t)\mathrm{d}t,$$

completing the result. □

Remark 1.34. *Notice that in the theorem above we do not require that the function f is invertible. A continuous function does map an interval into an interval and it is one-to-one if and only if it is monotone, which is not required in Theorem 1.33. We do not require that the function preserves the orientation of the interval, and indeed $f(a)$ could be greater than $f(b)$. If that is the case we understand the integral as an oriented integral and set (if $c > b$)*

$$\int_c^b f(x)\mathrm{d}x = -\int_b^c f(x)\mathrm{d}x, \tag{1.11}$$

that is sweeping the interval in the opposite direction reverses the value of the integral.

This is a natural definition. If $a < b < c$, using the additivity formula from Theorem 1.23 we have

$$\int_a^b f(x)\mathrm{d}x - \int_a^c f(x)\mathrm{d}x = -\left[\int_a^c f(x)\mathrm{d}x - \int_a^b f(x)\mathrm{d}x\right] = -\int_b^c f(x)\mathrm{d}x.$$

If we wanted to extend the additivity formula from Theorem 1.23 to apply to the case $a < b < c$ then we should have

$$\int_a^b f(x)\mathrm{d}x - \int_a^c f(x)\mathrm{d}x = \int_c^b f(x)\mathrm{d}x,$$

which requires the definition (1.11).

Remark 1.35. *In the proof of Theorem 1.33 we have used the chain rule to differentiate expressions of the form*

$$\int_a^{f(x)} g(t)\mathrm{d}t$$

for g continuous and f differentiable. Several examples will arise for example in MA250 - PDE, when considering $u(x,t) := \int_{x-ct}^{x+ct} \Psi(r)\mathrm{d}r$ as a solution of the wave equation, or the function $\int_0^{\frac{x}{\sqrt{4kt}}} \mathrm{e}^{-s^2}\mathrm{d}s$ when computing the fundamental solution of the heat equation. As this procedure will be fundamental we apply it in the next, more general example. Let $u(x,t) : \mathbb{R} \times \mathbb{R} \mapsto \mathbb{R}$ be given by (u could of course be defined in any open subset of $\mathbb{R}^2$ instead of the full $\mathbb{R}^2$)

$$u(x,t) := \int_{a(x,t)}^{b(x,t)} f(s)\mathrm{d}s$$

Then, assuming that a, b are differentiable and that f is continuous

$$\frac{\partial u}{\partial x}(x,t) = f(b(x,t))\frac{\partial b}{\partial x}(x,t) - f(a(x,t))\frac{\partial a}{\partial x}(x,t),$$

$$\frac{\partial u}{\partial t}(x,t) = f(b(x,t))\frac{\partial b}{\partial t}(x,t) - f(a(x,t))\frac{\partial a}{\partial t}(x,t).$$

Using integrals as a way to define new functions can be very useful in many applications. In fact, it can provide a simple way to prove properties of functions, as illustrated in this example.

Example 1.36. *We can define the logarithm using an integral. Namely we can set*

$$\ln x := \int_1^x \frac{1}{t}\mathrm{d}t.$$

As the function $\frac{1}{t}$ *is continuous on* $(0,\infty)$ *we now that* $\ln x$ *is well defined in that domain and that in fact* $(\ln x)' = \frac{1}{x}$*. Other properties of* $\ln$ *can be easily deduced as well.*

$$\ln x + \ln y = \int_1^x \frac{1}{t}\mathrm{d}t + \underbrace{\int_1^y \frac{1}{t}\mathrm{d}t}_{t=s/x} = \int_1^x \frac{1}{t}\mathrm{d}t + \int_x^{xy} \frac{1}{s}\mathrm{d}s = \int_1^{xy} \frac{1}{t}\mathrm{d}t = \ln(xy),$$

where we have used the change of variables $t = s/x$ *to obtain*

$$\int_1^y \frac{1}{t}\mathrm{d}t = \int_x^{xy} \frac{1}{s/x}\frac{1}{x}\mathrm{d}s = \int_x^{xy} \frac{1}{s}\mathrm{d}s.$$

1.4 Improper integrals

So far in this chapter we have considered Riemann integration, requiring bounded functions and a bounded domain of integration. In this section we extend the notion to include integrals of unbounded functions and/or in unbounded domains, via a limit procedure. This limit will be called an improper Riemann integral.

We start by consider the case in which the function f is unbounded at one of the endpoints.

Definition 1.37. *Let* $f : [a,b] \to \mathbb{R}$ *be a Riemann integrable function for every* $[c,b]$ *with* $a < c$*. Then the improper integral of* f *on* $[a,b]$ *is defined as*

$$\int_a^b f(x)\mathrm{d}x = \lim_{\varepsilon\to 0^+} \int_{a+\varepsilon}^b f(x)\mathrm{d}x.$$

The improper integral $\int_a^b f(x)\mathrm{d}x$ *is said to converge if the limit is finite. Otherwise it is divergent. Similarly, for a function that is unbounded at* b *and integrable on all* $[a,c]$ *with* $c < b$ *we define*

$$\int_a^b f(x)\mathrm{d}x = \lim_{\varepsilon\to 0^+} \int_a^{b-\varepsilon} f(x)\mathrm{d}x.$$

With this definition it is possible to revisit the integrals

$$\int_0^1 x^p\mathrm{d}x.$$

For $-1 < p \le 0$ we have (using antiderivatives once we are in the interval $[\varepsilon,1]$)

$$\int_0^1 x^p\mathrm{d}x = \lim_{\varepsilon\to 0^+} \int_\varepsilon^1 x^p\mathrm{d}x = \lim_{\varepsilon\to 0^+} \frac{1}{p+1} - \frac{1}{p+1}\varepsilon^{p+1} = \frac{1}{p+1}.$$

In the case in which $p = -1$ we have

$$\int_0^1 x^{-1}\mathrm{d}x = \lim_{\varepsilon\to 0^+} \int_\varepsilon^1 x^{-1}\mathrm{d}x = \lim_{\varepsilon\to 0^+} (-\ln\varepsilon) = \infty,$$

and so the integral is divergent.

We can also consider improper integrals for functions that are unbounded near an interior point c.

Definition 1.38. *Let* $f : [a,b] \to \mathbb{R}$ *be a function that is integrable on any closed interval not containing* $c \in [a,b]$*, that is on all* $[a, c-\varepsilon]$ *and* $[c+\delta, b]$*, for* $\varepsilon,\delta > 0$ *sufficiently small. Then we define the improper integral of* f *on* $[a,b]$

$$\int_a^b f(x)\mathrm{d}x = \lim_{\varepsilon\to 0^+} \int_a^{c-\varepsilon} f(x)\mathrm{d}x + \lim_{\delta\to 0^+} \int_{c+\delta}^b f(x)\mathrm{d}x.$$

We can also define improper integrals for unbounded domains by taking limits of bounded intervals.

Definition 1.39. *Let $f : [a, \infty) \to \mathbb{R}$ be a function that is integrable for every interval $[a, y]$, for $a < y < \infty$. We define the improper integral of f on $[a, \infty]$ by*

$$\int_a^\infty f(x)\mathrm{d}x = \lim_{y\to\infty} \int_a^y f(x)\mathrm{d}x.$$

Similarly if $g : (-\infty, b] \to \mathbb{R}$ is an integrable function for every interval $[y, b]$ with $-\infty < y < b$, then we define the improper integral of g on $(-\infty, b]$ by

$$\int_{-\infty}^b g(x)\mathrm{d}x = \lim_{y\to-\infty} \int_y^b g(x)\mathrm{d}x.$$

Definition 1.40. *Let $f : \mathbb{R} \to \mathbb{R}$ be a function that is integrable on every bounded interval $[a, b]$. Then we define the improper integral*

$$\int_{-\infty}^\infty f(x)\mathrm{d}x = \lim_{a\to-\infty} \int_a^c f(x)\mathrm{d}x + \lim_{b\to\infty} \int_c^b f(x)\mathrm{d}x,$$

with c any point in $\mathbb{R}$.

Notice that in the definition above we require that the improper integrals $\int_{-\infty}^c f$ and $\int_c^\infty$ are finite, rather than considering a single limit as a goes to ∞ of $\int_{-a}^a f(x)\mathrm{d}x$. It is clear that the function $f(x) = x$ is not integrable in $\mathbb{R}$ in the sense of Definition 1.40, but since $\int_{-a}^a x\mathrm{d}x = 0$ we would have obtained that the integral is zero.

Remark 1.41. *Improper Riemann integrable functions form a linear space. That is, if f and g are improperly integrable on the same domain (bounded or unbounded) then so is $\alpha f + \beta g$ for any $\alpha, \beta \in \mathbb{R}$.*

Remark 1.42. *Several of the results we have seen before are not true for improper integrals. In particular the product of two improperly integrable functions might not be integrable. As an example, consider $f(x) = \frac{1}{\sqrt{x}}$ on $[0, 1]$. We have seen that it is improperly integrable. However f^2 equals $\frac{1}{x}$ which is not integrable. Similarly the composition of a continuous function with an improperly Riemann integrable function need not be integrable. As an example, consider $g(x) = x^2$ on $[0, 1]$ and $f(x) = \frac{1}{\sqrt{x}}$ on $[0, 1]$. Now, $g \circ f(x) = \frac{1}{x}$ on $[0, 1]$, which is not integrable. Additionally, if f is improperly Riemann integrable then $|f|$ need not be. For example, on $[0, \infty)$ the function*

$$f(x) = \sum_{n=1}^\infty (-1)^n \frac{1}{n} \chi_{[n,n+1)}(x), \qquad \text{with } \chi_{[n,n+1)}(x) = 1 \text{ if } x \in [n, n+1) \quad \& \quad 0 \text{ otherwise}$$

is improperly integrable as the alternating series $\sum_{n=1}^\infty (-1)^n \frac{1}{n}$ is finite. However $|f|$ is not integrable as the series $\sum_{n=1}^\infty \frac{1}{n} = \infty$.

1.5 The Cantor Set and the devil's staircase

In this section we will construct the standard Cantor set. This is a set with many interesting features and will allow us to build many counterexamples to different results.

The Cantor set is a subset of $[0, 1]$ constructed iteratively. We set $C_0 = [0, 1]$. In the next stage we remove the open interval $(1/3, 2/3)$ and set $C_1 = [0, 1/3] \cup [2/3, 1]$. We proceed inductively. Each C_k is a finite union of 2^k closed intervals of length $1/3^k$. To construct C_{k+1} we remove, from each of the intervals in the set C_k the corresponding open middle third. (For the second iteration that corresponds to $C_2 = [0, 1/9] \cup [2/9, 1/3] \cup [2/3 \cup 7/9] \cup [8/9, 1]$.) It is clear that the intervals removed, and those remaining have length $1/3^{k+1}$. Also, as indicated we have 2^{k+1} intervals remaining.

Notice that the sets C_k form a nested sequence, with $C_k \supset C_{k+1}$ and that we can define the Cantor set C as the limit of those C_k,

$$C := \lim C_k = \cap C_k.$$

The set is clearly not empty since at least the endpoints of the intervals in each C_k remain.

Notice that at stage k we remove 2^{k-1} intervals (one for each interval in the previous generation C_{k-1}) of length $1/3^k$. Therefore the total length we have removed is

$$\sum_{k=1}^{\infty} \frac{2^{k-1}}{3^k} = \frac{1}{3}\sum_{k=1}^{\infty}\left(\frac{2}{3}\right)^{k-1} = \frac{1}{3}\sum_{k=0}^{\infty}\left(\frac{2}{3}\right)^{k} = \frac{1}{3}\frac{1}{1-\frac{2}{3}} = 1.$$

Therefore the length, of C, or its measure (a notion that we have not defined precisely) is zero. However the Cantor set has the cardinality of $\mathbb{R}$. To see this, we can think of every $x \in [0,1]$ as written in base 3, with an expansion of the form $0.a_1a_2a_3\dots$ in which a_i can take the values 0, 1 or 2. This is equivalent to representing x as $\sum_{k=1}^{\infty} \frac{a_k}{3^k}$. Notice that this description of x might not be unique. Indeed $1/3$ can represented as 0.1 as well as $0.0\overline{2}$ periodic.

We claim that the Cantor set is the collections of points that do not contain any 1 in their expansion. If we consider C_1, we removed the interval $(1/3, 2/3)$, i.e. every x for which $a_1 = 1$ (this is perhaps more easily seen with the series representation of x). It would appear that if we do this we also remove the point $1/3$. However since that point has two representations, it still remains in C_1 as $0.0\overline{2}$. It is easy to see that this is what happens for all iterations and that this is an equivalent way of describing C.

We can construct a surjective map from C to $[0,1]$ that proves that C has the same cardinality as $[0,1]$. Set

$$x = \sum_{k=1}^{\infty} \frac{a_k}{3^k} \mapsto f(x) = \sum_{k=1}^{\infty} \frac{\frac{a_k}{2}}{2^k}.$$

First notice that $a_k/2$ is either 0 or 1 given that x is in C. If we think of $f(x)$ as the binary expression of a number between $[0,1]$ we see that the range of f is the entire set $[0,1]$.

The Cantor set has many other properties that we do explore here. It is self-similar, it is perfect (i.e. C is equal to the set of all limit points of C), it is totally disconnected, nowhere dense (the interior of the closure of the set is empty), ...

We now construct the Devil's staircase function.

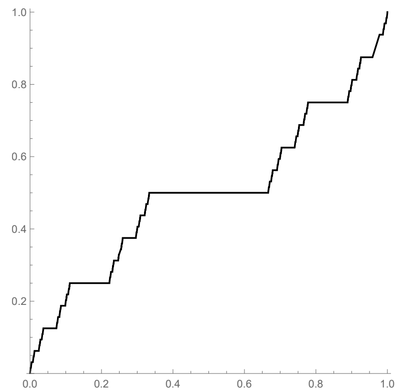

Figure 1.4: The Devil's staircase.

This is built out of the Cantor set and is a continuous non-decreasing function from $[0,1]$ to $[0,1]$ that is constant on a set of measure 1. Again the construction is via an iteration argument. Consider $f_0 : [0,1] \to [0,1]$ given by $f_0(x) = x$. It is clearly continuous, nondecreasing. Now we construct a function in C_1, given by

$$f_1(x) = \begin{cases} \frac{3}{2}x & x \in [0, \frac{1}{3}] \\ \frac{1}{2} & x \in [\frac{1}{3}, \frac{2}{3}] \\ \frac{1}{2} + \frac{3}{2}(x - \frac{2}{3}) & x \in [\frac{2}{3}, 1] \end{cases}$$

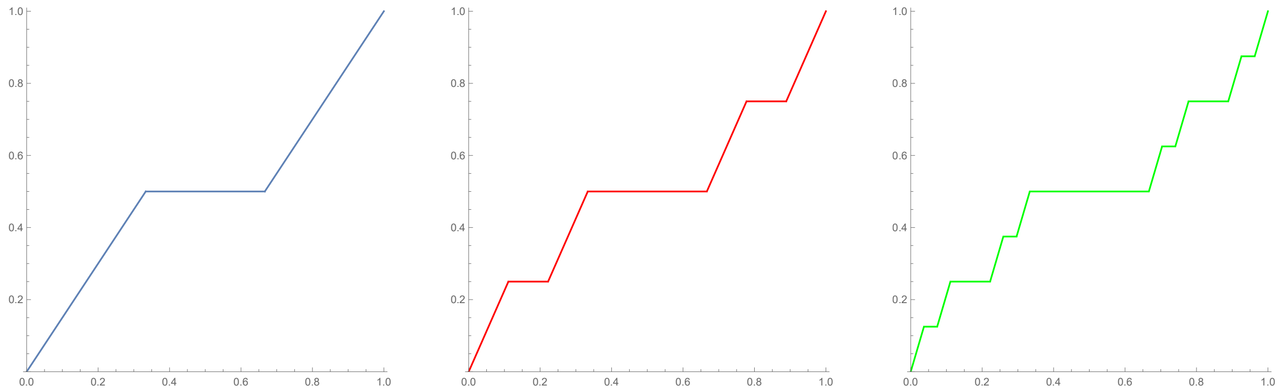

Figure 1.5: First 3 iterates in the construction of the Devil's staircase

To construct f_2 we consider the intervals in which f_1 has nonzero slope (the intervals left in C_1 in the iterative construction of C). In each of those intervals we insert a scaled down version of f_1 as indicated in Figure 1.5. The third iterate proceeds similarly by inserting a scaled down version of f_1 in each interval in which f_2 has nonzero slope. An alternative way of thinking about this is as follows. Contruct f_{n+1} by keeping it constant in the middle interval $(1/3, 2/3)$, and by inserting in the first and third a scale down copy of f_n. More precisely, set $f_0(x) = x$ as before and define

$$f_{n+1}(x) = \begin{cases} \frac{1}{2} f_n(3x) & x \in [0, 1/3) \\ \frac{1}{2} & x \in [\frac{1}{3}, \frac{2}{3}) \\ \frac{1}{2} + \frac{1}{2} f_n(3x-2) & x \in [\frac{2}{3}, 1]. \end{cases}$$

It is easy to check that both procedures lead to the same function.

Theorem 1.43. *The limit of the sequence* (f_n) *exists and is continuous with* $f(0) = 0$ *and* $f(1) = 1$.

Proof. Notice that f_0 is trivially continuous and as a result all the f_n are trivially continuous except for at $\frac{1}{3}$ and $\frac{2}{3}$. It is easy to check the two limits (from above and from below) arising from the different regions of definition coincide and that therefore f_n is continuous. (Left as an exercise.) We will prove that (f_n) converges uniformly, from which the continuity will follow. Since $f_n(0) = 1$ and $f_n(1) = 1$ for every n the rest of the Theorem follows.

We will prove the uniform convergence by proving that the sequence is uniformly Cauchy. We start by considering the distance between two consecutive iterates

$$\begin{aligned} \|f_{n+1} - f_n\|_\infty &= \max(\left\|(f_{n+1} - f_n)|_{x\in[0,1/3)}\right\|_\infty, \left\|(f_{n+1} - f_n)|_{x\in[1/3,2/3)}\right\|_\infty, \left\|(f_{n+1} - f_n)|_{x\in[2/3,1)}\right\|_\infty) \\ &= \max(\left\|\frac{1}{2} f_n(3x) - \frac{1}{2} f_{n-1}(3x)|_{x\in[0,1/3)}\right\|_\infty, 0, \left\|\frac{1}{2} f_n(3x-2) - \frac{1}{2} f_{n-1}(3x-2)|_{x\in[2/3,1)}\right\|_\infty) \\ &= \frac{1}{2}\|f_n - f_{n-1}\|_\infty. \end{aligned}$$

Therefore

$$\|f_{n+1} - f_n\|_\infty = \frac{1}{2}\|f_n - f_{n-1}\|_\infty = \frac{1}{2^2}\|f_{n-1} - f_{n-2}\|_\infty = \cdots = \frac{1}{2^n}\|f_1 - f_0\|_\infty = \frac{1/6}{2^n},$$

where we have used $\|f_1 - f_0\|_\infty = 1/6$, achieved at $1/3$ and $2/3$. To prove that (f_n) is Cauchy we use a telescopic expansion and use the estimate above. We have

$$\begin{aligned} \|f_n - f_m\|_\infty &= \|f_n - f_{n-1} + f_{n-1} - f_{n-2} + \cdots + f_{m+1} - f_m\|_\infty \\ &\le \|f_n - f_{n-1}\|_\infty + \|f_{n-1} - f_{n-2}\|_\infty + \cdots + \|f_{m+1} - f_m\|_\infty \end{aligned}$$

$$\leq \sum_{k=m}^{n-1} \|f_{k+1} - f_k\|_\infty = \frac{1}{6} \sum_{k=m}^{n-1} \frac{1}{2^{k-1}}.$$

Now, since $\sum_{k=1}^{\infty} \frac{1}{2^{k-1}}$ is convergent we can make $\sum_{k=m}^{n-1} \dfrac{1}{2^{k-1}}$ arbitrarily small by choosing m, n large enough, proving that the sequence is uniformly Cauchy.

□

Chapter 2

Sequences and Series of Functions

In this Chapter we will consider sequences and series of functions and aspects relating to pointwise and uniform convergence and its interactions with continuity, integrability and differentiability questions.

2.1 Pointwise and uniform convergence

We will consider sequences of functions $f_n : \Omega \to \mathbb{R}$ from a fixed domain Ω. Here we do not make any assumptions about Ω, i.e. being open or closed, bounded or unbounded for example. While most examples will be in one dimension, unless otherwise noted they apply to higher dimensions. We start by defining pointwise convergence.

Definition 2.1. *Let $(f_n)_{n=1}^{\infty}$ be a sequence of functions, with $f_n : \Omega \to \mathbb{R}$. We say that (f_n) or f_n converges pointwise to $f : \Omega \to \mathbb{R}$ if and only if for every $x \in \Omega$ we have $\lim_{n\to\infty} f_n(x) = f(x)$. We will denote pointwise convergence by $f_n \to f$.*

Example 2.2. *Consider the sequence (f_n) given by $f_n : [0,1] \to \mathbb{R}$, $f_n(x) = x^{1/n}$.*

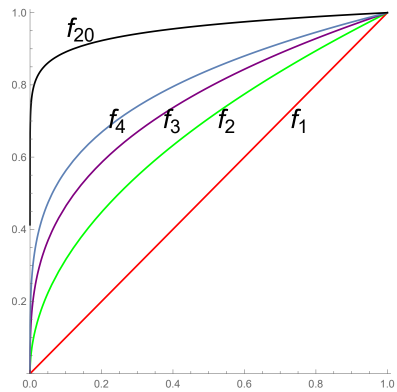

Figure 2.1: The sequence f_n for $n = 1, 2, 3, 4$ and 20.

Notice that $f_n(0) = 0$ for every n, but that for every $x \in (0,1]$ we have $\lim_{n\to\infty} x^{1/n} = 1$. As a result the limit of the sequence (f_n) is

$$f(x) = \begin{cases} 0 & x = 0, \\ 1 & x \in (0,1]. \end{cases}$$

Remark 2.3. *Notice that the above example shows that the pointwise limit of a sequence of continuous functions need not be continuous. It also produces a counterexample for the commutativity of the limits. We have*

$$\lim_{n\to\infty} \lim_{x\to 0^+} f_n(x) \neq \lim_{x\to 0^+} \lim_{n\to\infty} f_n(x),$$

as the left-hand side equals zero, while the right-hand side equals one.

Pointwise convergence clearly does not preserve continuity. It can also be very non-uniform, in the sense that while $f_n(x) \to 0$ for every x we may have $\sup_x |f_n(x) - f(x)| \to C > 0$ or even $\sup_x |f_n(x) - f(x)| \to \infty$ as n goes to infinity, as shown in the next examples.

Example 2.4. *Consider the sequences*

$$g_n(x) = \begin{cases} 2nx & x \in [0, \frac{1}{2n}) \\ -2n(x - \frac{1}{n}) & x \in [\frac{1}{2n}, \frac{1}{n}) \\ 0 & x \in [\frac{1}{n}, 1] \end{cases} \qquad h_n(x) = \begin{cases} 2n^2x & x \in [0, \frac{1}{2n}) \\ -2n^2(x - \frac{1}{n}) & x \in [\frac{1}{2n}, \frac{1}{n}) \\ 0 & x \in [\frac{1}{n}, 1]. \end{cases}$$

It is easy to see that g_n and h_n are continuous and converge to the function $f = 0$. However, for every n we have $g_n(1/(2n)) = 1$ (with that being the maximum of g_n) and therefore

$$\sup_{x\in[0,1]} |g_n(x) - 0| = 1.$$

The situation is worse for the sequence (h_n), known as the Witch's hat. Indeed $h(1/(2n)) = n$, which shows that while $h_n \to 0$ we have

$$\sup_{x\in[0,1]} |h_n(x) - 0| \to \infty.$$

Pointwise convergence and integrability do not interact as one would hope. Indeed, even if we assume that the pointwise limit is integrable we may not have $\lim \int f_n = \int \lim f_n$.

Example 2.5. *Consider $f_n(x) = \chi_{[n,n+1)}(x)$, where χ_I is the indicator of the set I, i.e., takes value 1 if $x \in I$ and zero otherwise. Clearly f_n converges pointwise to $f = 0$. However,*

$$1 = \int f_n \neq \int f = 0.$$

We can think of this, as "the mass scaping to infinity" (along the x axis).

Another example of this phenomena, can be found by considering $g_n(x) = n\chi_{(0,1/n)}(x)$ we also have that g_n converges to 0, while having $\int g_n = 1$ for every n. We can think of this as "pointwise convergence allowing the mass to go to infinity" (along the y axis this time). The Witch's hat above also provides a similar example, in this case with continuous functions.

Example 2.6. *Another sequence that will play a role in several modules this year is $f_n(x) = \sin(n\,x)$. This sequence is connected to Fourier series and will be heavily studied in MA250 PDE for example. Notice that for $x = k\pi$ with $k \in \mathbb{Z}$ the limit exists and equals 0. If $x = p/q\,\pi$ with $p/q \notin \mathbb{Z}$ then there is no limit. Indeed $\sin(nqx) = 0$ while $\sin((2nq + 1)x) = \sin(x) \neq 0$. If x is an irrational multiple of π, then the rest of the division of nx by 2π is dense in $[0, 2\pi]$ and there is no limit.*

Despite the fact that $\sin(nx)$ does not have a limit for most x, you will see in MA250 that for every integrable function f

$$\int_{-\pi}^{\pi} f(x)\sin(nx)\mathrm{d}x \to 0 \qquad \text{as } n \to \infty.$$

This result, known as the Riemann–Lebesgue Lemma, suggests that $\sin(nx)$ goes to zero in some sense (known as we the weak sense, which will be covered in Measure Theory, Functional Analysis and Fourier Analysis). We can also consider the sequence $g_n(x) = \frac{\cos(nx)}{n}$. As cosine is a bounded function it is easy to see that g_n converges pointwise to 0. Since g_n are smooth we can also consider $g_n'(x) = -\sin(nx)$. This tells us that even for smooth functions, having g_n converge pointwise to g does not imply that g_n' converges to g' even if g is smooth.

The final example we consider is one of a sequence (f_n) such that $\int(f_n - f)\mathrm{d}x$ converges to zero, but where f_n does not converge pointwise to f.

Example 2.7. *We will consider functions defined on* $[0,1]$*. Let*

$$f_0(x) = \chi_{[0,1]}(x),$$

$$f_1(x) = \chi_{[0,1/2]}, \qquad f_2(x) = \chi_{[1/2,1]},$$

$$f_3(x) = \chi_{[0,1/4]}, \qquad f_4(x) = \chi_{[1/4,1/2]}, \qquad f_5(x) = \chi_{[1/2,3/4]}, \qquad f_6(x) = \chi_{[3/4,1]}.$$

Notice that each function is an indicator of an interval, and that in each group above the intervals sweep $[0,1]$*. When we move to the next block the length of the corresponding intervals gets divided by 2 and therefore we consider twice as many functions for each group. While the process is clear from the list writing a formula for* f_n *is annoying to say the least. You can check that the following works. For an index*

$$n \in [\sum_{l=0}^{k-1} 2^l, \sum_{l=0}^{k} 2^l], \qquad k = 1, 2, \dots$$

we set f_n *as the indicator of the interval*

$$\left[\frac{n - \sum_{l=0}^{k-1} 2^l}{2^k}, \frac{n - \sum_{l=0}^{k-1} 2^l + 1}{2^k}\right].$$

Since the length of the intervals tends to zero it is clear that $\int f_n \to 0$*, but since the intervals keep sweeping the entire interval* $[0,1]$ *the sequence* f_n *does not converge to zero (or any other function for that matter). This is contrary to the intuition that if the area between* f *and* f_n *is going to zero the functions* f_n *must be approaching, and therefore converging to* f*.*

We now consider the notion of uniform convergence.

Definition 2.8. *Let* $f_n : \Omega \to \mathbb{R}$ *be a sequence of functions. We say that* (f_n) *converges uniformly to* $f : \Omega \to \mathbb{R}$ *if and only if for every* $\varepsilon > 0$ *there exists* $N(\varepsilon)$ *such that* $|f_n(x) - f(x)| < \varepsilon$ *for every* $x \in \Omega$ *and for all* $n > N(\varepsilon)$*.*

The key different with pointwise convergence is that N depends only on ε and not on x. For pointwise convergence we first froze x and consider the convergence of $f_n(x)$ to $f(x)$. We will denote uniform convergence by $f_n \rightrightarrows f$.

As before we are not making any assumption on Ω. In order to simplify the presentation we introduce the notation

$$\|f\|_\infty = \sup_{x\in\Omega} |f(x)|.$$

With this notation we have

$$f_n \rightrightarrows f \iff \forall\varepsilon > 0,\ \exists N(\varepsilon) \text{ such that } \|f_n - f\|_\infty < \varepsilon\ \forall n > N(\varepsilon).$$

Remark 2.9. *Clearly uniform convergence implies pointwise convergence. The converse is of course false, as we have seen in Remark 2.3.*

Definition 2.10. *A sequence* (f_n) *of functions in* Ω *is called uniformly Cauchy if and only if for every* $\varepsilon > 0$ *there exists* $N(\varepsilon)$ *such that* $\|f_n - f_m\|_\infty < \varepsilon$ *for all* $n, m > N(\varepsilon)$ *(or alternatively* $|f_n(x) - f_m(x)| < \varepsilon$ *for every* $x \in \Omega$ *and all* $n, m > N(\varepsilon)$*).*

Theorem 2.11. *A sequence* (f_n) *is uniformly convergent if and only if it is uniformly Cauchy.*

Proof. Assume that (f_n) is uniformly convergent to f, i.e. for every ε there exists N such that $\|f_n - f\|_\infty < \varepsilon/2$. Then

$$\|f_n - f_m\|_\infty \le \|f_n - f + f - f_m\|_\infty \le \|f_n - f\|_\infty + \|f_m - f\|_\infty \le \varepsilon/2 + \varepsilon/2 = \varepsilon.$$

For the converse, assume (f_n) is uniformly Cauchy. That means that for every x, $f_n(x)$ is a Cauchy sequence in $\mathbb{R}$ and therefore convergent. That means there exists $f(x)$ such that $f_n(x)$ converges to $f(x)$ at least pointwise. Now, we know that given $\varepsilon > 0$ there exists $N(\varepsilon) > 0$ such that $|f_n(x) - f_m(x)| < \varepsilon$ for every x and all $n, m > N(\varepsilon)$. That is

$$f_m(x) - \varepsilon < f_n(x) < f_m(x) + \varepsilon \text{ for all } x, \quad \text{and all } n, m > N(\varepsilon).$$

As the left-hand side holds for all $m > N(\varepsilon)$ we can take limits as m goes to infinity. We find

$$f(x) - \varepsilon \leq f_n(x) \leq f(x) + \varepsilon \text{ for all } x, \quad \text{and all } n > N(\varepsilon).$$

from which it follows that

$$|f(x) - f_n(x)| < 2\varepsilon \text{ for all } x, \quad \text{and all } n > N(\varepsilon),$$

which proves the result. □

Remark 2.12. *While this topic will be discussed in more depth in Norms, Metrics and Topologies it is worth noting that $\|\cdot\|_\infty$ is a norm in the space of bounded functions in Ω (we make no assumptions about it being open, closed, bounded or unbounded). $\|\cdot\|_\infty$ is referred to as the supremum norm. Recall that by norm we mean that it satisfies*

1. $\|f\|_\infty \geq 0$, *with* $\|f\|_\infty = 0$ *if and only if* $f = 0$,
2. $\|\lambda f\|_\infty = |\lambda| \|f\|_\infty$, *for all* $\lambda \in \mathbb{R}$, *and*
3. $\|f + g\|_\infty \leq \|f\|_\infty + \|g\|_\infty$.

Theorem 2.13. *Let (f_n) be a sequence of continuous functions in Ω that converges uniformly to $f : \Omega \to \mathbb{R}$. Then f is continuous.*

Proof. First notice that the uniform convergence implies that given any $\varepsilon > 0$ there exists $N > 0$ such that $\|f_n - f\|_\infty < \varepsilon/3$ for all $n > N$. In order to show that f is continuous at $x_0 \in \Omega$ we need to show that given ε there exists $\delta = \delta(\varepsilon)$ such that for all $x \in (x_0 - \delta, x_0 + \delta) \cap \Omega$ we have $|f(x) - f(x_0)| < \varepsilon$. With N as above, we choose $n > N$, fixed from now own. Since f_n is continuous at x_0 we know that there exists $\delta = \delta(\varepsilon)$ such that for all $x \in (x_0 - \delta, x_0 + \delta) \cap \Omega$ we have $|f_n(x) - f_n(x_0)| < \varepsilon/3$.

We estimate $|f(x) - f(x_0)|$ using the triangle inequality

$$\begin{aligned} |f(x) - f(x_0)| &= |f(x) - f_n(x) + f_n(x) - f_n(x_0) + f_n(x_0) - f(x_0)| \\ &\leq |f(x) - f_n(x)| + |f_n(x) - f_n(x_0)| + |f_n(x_0) - f(x_0)| \\ &\leq \|f_n - f\|_\infty + |f_n(x) - f_n(x_0)| + \|f_n - f\|_\infty \leq \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3}, \end{aligned}$$

for $n > N$ and $x \in (x_0 - \delta, x_0 + \delta) \cap \Omega$, with N and δ chosen as above. This completes the proof. □

We will denote the space of bounded, continuous functions with the uniform norm by $(C_b; \|\cdot\|_\infty)$.

Theorem 2.14. *$(C_b; \|\cdot\|_\infty)$ is a complete space, i.e. every Cauchy sequence converges to a continuous bounded function.*

Proof. We need to show that if (f_n) is Cauchy in the space, then there is a limit, and that the limit is bounded and continuous. First notice that a Cauchy sequence in $(C_b; \|\cdot\|_\infty)$ is, by definition, a uniformly Cauchy sequence. Theorem 2.11 implies that the sequence is convergent and since all the functions are continuous Theorem 2.13 implies the limit is continuous.

To see that it is bounded, notice that for every $x \in \Omega$

$$|f(x)| \leq |f(x) - f_n(x)| + |f_n(x)|$$

for every n. Since f_n converges uniformly to f there exists n large enough $|f_n(x) - f(x)| < 1$. For that n, since f_n is bounded we have $|f_n| \leq M$. These two inequalities lead to $|f(x)| \leq M + 1$ for every $x \in \Omega$, proving the boundedness of f. □

Remark 2.15. *We could consider the interaction of uniform convergence and differentiation or integration. Consider for example $f_n(x) = \frac{\sin(n^2x)}{n}$. The sequence (f_n) converges to $f = 0$ uniformly. Indeed*

$$\left|\frac{\sin(n^2x)}{n} - 0\right| \le \frac{1}{n} \qquad \forall x.$$

Clearly all the functions f_n are smooth. The derivatives are given by $f_n'(x) = n\cos(n^2x)$. It is easy to see that the sequence (f_n') does not converge uniformly (or pointwise). This example shows that while $f_n \rightrightarrows f$ we may not have $f_n' \rightrightarrows f'$ or even $f_n' \to f'$.

To explore integrability, we consider $g_n(x) = \frac{1}{2n}\chi[-n,n]$. Recall that strictly speaking we have not defined Riemann integration in $\mathbb{R}$, but rather improper integration, via a limiting procedure. It is clear however that $\int g_n = 1$ for every n. The sequence g_n converges uniformly to $g = 0$ as we have $|g_n - 0| \le 1/(2n)$, and so $\lim \int g_n = 1 \ne 0 = \int g$. We reiterate that strictly speaking g_n are not Riemann integrable and we will prove that in fact, on a bounded interval $f_n \rightrightarrows f$ does imply $\int f_n \to \int f$.

Theorem 2.16. *Lef (f_n), $f_n : [a,b] \to \mathbb{R}$ be a sequence in Riemann integrable functions that converges uniformly to $f : [a,b] \to \mathbb{R}$. Then f is Riemann integrable and $\int f_n \to \int f$.*

Proof. First we need to show that f is Riemann integrable, that is show that for every $\varepsilon > 0$ there exists a partition P of $[a,b]$ such that

$$U(f,P) - L(f,P) < \varepsilon.$$

Now, since $f_n \rightrightarrows f$ we know that for any $\varepsilon > 0$ there exists N such that $\|f_n - f\|_\infty < \varepsilon/(4(b-a))$ for $n > N$. For a fixed $n > N$ since f_n is integrable we know that given $\varepsilon > 0$ there exists a partition P such that

$$U(f_n,P) - L(f_n,P) < \frac{\varepsilon}{2}.$$

Now, for that P

$$\begin{aligned}
U(f,P) - L(f,P) &= \sum[\sup_{I_k} f - \inf_{I_k} f]|I_k| = \sum[\sup_{I_k}(f - f_n + f_n) - \inf_{I_k}(f - f_n + f_n)]|I_k| \\
&\le \sum\left[\|f - f_n\|_\infty + \sup_{I_k} f_n + \|f - f_n\|_\infty - \inf_{I_k} f_n\right]|I_k| \\
&= 2\sum\|f - f_n\|_\infty |I_k| + \sum[\sup_{I_k} f_n - \inf_{I_k} f_n]|I_k| \\
&\le 2\|f - f_n\|_\infty(b-a) + U(f_n,P) - L(f_n,P) \\
&\le 2\frac{\varepsilon}{4(b-a)}(b-a) + \frac{\varepsilon}{2} = \varepsilon.
\end{aligned}$$

To see that $\int f_n \to \int f$, notice that

$$\left|\int_a^b f_n - \int_a^b f\right| \le \left|\int_a^b f_n - f\right| \le \int_a^b |f_n - f| \le \int_a^b \|f - f_n\|_\infty = \|f_n - f\|_\infty(b-a).$$

Clearly the right hand side goes to zero as n goes to infinity by the uniform convergence of (f_n) to f. □

In many circumstances it is necessary to consider functions of two variables (or more) from which we construct new functions by integrating out some of the variables. We want to study several results in this direction; we start by reviewing the notions of continuity and uniform continuity in two dimension. The definitions are analogous to Definitions 1.12 and 1.13.

Definition 2.17. *Given $f : \Omega \subset \mathbb{R}^2 \to \mathbb{R}$, we say that f is continuous at x if for every $\varepsilon > 0$ there exists $\delta = \delta(x, \varepsilon) > 0$ such that*

$$y \in \Omega \text{ and } |x - y| < \delta \Longrightarrow |f(y) - f(x)| < \varepsilon. \tag{2.1}$$

Note that $|\cdot|$ has been used both to denote Euclidean distance in the plane, as in $|x - y|$, as well as for absolute value of a real number, in $|f(y) - f(x)|$.

Definition 2.18. *Given $f : \Omega \subset \mathbb{R}^2 \to \mathbb{R}$, we say that it is uniformly continuous if for every $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon) > 0$ such that*

$$x, y \in \Omega \text{ and } |x - y| < \delta \Longrightarrow |f(y) - f(x)| < \varepsilon. \tag{2.2}$$

The key point here is that δ can be chosen independently of x. Similarly to Theorem 1.14 we have the following result.

Theorem 2.19. *Let $f : \Omega \subset \mathbb{R}^2 \to \mathbb{R}$ be a continuous function. Assume that Ω is closed and bounded. Then it is uniformly continuous.*

Proof. We will argue by contradiction. That would mean that there exists $\varepsilon > 0$ and x_n, y_n such that $|x_n - y_n| \leq \frac{1}{n}$ but $|f(x_n) - f(y_n)| > \varepsilon$.

The sequences $\{x_n\}$ and $\{y_n\}$ are bounded, as they are in Ω, which is closed and bounded, and therefore we can apply Bolzano–Weierstrass to each component to obtain convergent subsequences $\{x_{n_k}\}_{k=1}^{\infty}$ to x and $\{y_{n_k}\}_{k=1}^{\infty}$ to y. Notice that

$$|x - y_{n_k}| \leq |x - x_{n_k}| + |x_{n_k} - y_{n_k}| \leq |x - x_{n_k}| + \frac{1}{n_k} \xrightarrow[k\to\infty]{} 0,$$

which implies that $x = y$. However we know that $|f(x_{n_k}) - f(y_{n_k})| > \varepsilon$ for all k. Since f is continuous, taking limits as k goes to infinity we obtain $0 = |f(x) - f(x)| > \varepsilon$, which is a contradiction. □

Theorem 2.20. *Let $f : [a, b] \times [c, d] \to \mathbb{R}$ be a continuous function. Define*

$$I(t) := \int_a^b f(x, t)\mathrm{d}x$$

Then I is a continuous function on $[c, d]$.

Proof. We need to show that for every $\varepsilon > 0$ there exists δ such that $|t - t_0| < \delta$, and $t, t_0 \in [c, d]$ implies $|I(t) - I(t_0)| < \varepsilon$.

Now $I(t) - I(t_0) = \int [f(x, t) - f(x, t_0)]\mathrm{d}x$ and therefore

$$|I(t) - I(t_0)| \leq \int |f(x, t) - f(x, t_0)|\mathrm{d}x. \tag{2.3}$$

Since f is continuous on $[a, b] \times [c, d]$ it is uniformly continuous, and therefore given $\varepsilon > 0$ there exists δ such that $(x_1, t_1), (x_2, t_2) \in [a, b] \times [c, d]$ with $\sqrt{(x_1 - x_2)^2 + (t_1 - t_2)^2} < \delta$ implies that $|f(x_1, t_1) - f(x_2, t_2)| < \varepsilon/(b - a)$. Therefore if $|t - t_0| < \delta$ we have $|f(x, t) - f(x, t_0)| < \varepsilon/(b - a)$. As a result (2.3) becomes

$$|I(t) - I(t_0)| \leq \int |f(x, t) - f(x, t_0)|\mathrm{d}x < \int_a^b \frac{\varepsilon}{b - a}\mathrm{d}x = \varepsilon,$$

and we obtain the desired result. □

We can also consider differentiating I with respect to t under sufficient regularity results for f.

Theorem 2.21. *Lef $f, \frac{\partial f}{\partial t}$ be continuous functions on $[a, b] \times [c, d]$. Then, for $t \in (c, d)$*

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_a^b f(x, t)\mathrm{d}x = \int_a^b \frac{\partial f}{\partial t}(x, t)\mathrm{d}x.$$

Proof. Set $F(t) := \int_a^b f(x,t)\mathrm{d}x$, and $G(t) := \int_a^b \frac{\partial f}{\partial t}(x,t)\mathrm{d}x$. We want to show that $F' = G$. We consider the difference between the incremental quotient that is used to define a derivative of F and the function we expect to be derivative, namely G. For h sufficiently small so that $t+h \in [c,d]$

$$\left|\frac{F(t+h)-F(t)}{h} - G(t)\right| = \left|\int_a^b \frac{f(x,t+h)-f(x,t)}{h} - \frac{\partial f}{\partial t}(x,t)\mathrm{d}x\right|,$$

which by the Mean Value Theorem, becomes, for some $\tau \in (t, t+h)$

$$= \left|\int_a^b \frac{\partial f}{\partial t}(x,\tau) - \frac{\partial f}{\partial t}(x,t)\mathrm{d}x\right| \leq \int_a^b \left|\frac{\partial f}{\partial t}(x,\tau) - \frac{\partial f}{\partial t}(x,t)\right| \mathrm{d}x.$$

Now, since $\frac{\partial f}{\partial t}$ is continuous on $[a,b]\times[c,d]$ it is uniformly continuous, and therefore for every $\varepsilon > 0$ there exists δ such that for $|h| < \delta$ we have

$$\left|\frac{\partial f}{\partial t}(x,\tau) - \frac{\partial f}{\partial t}(x,t)\right| < \frac{\varepsilon}{b-a}$$

This is implies that for $|h| < \delta$

$$\left|\frac{F(t+h)-F(t)}{h} - G(t)\right| \leq \int_a^b \frac{\varepsilon}{b-a} = \varepsilon.$$

Taking limits as h goes to zero we have

$$|F'(t) - G(t)| \leq \varepsilon,$$

and since ε is arbitrary we are done. □

We now explore a version of Fubini's Theorem for continuous functions.

Theorem 2.22. *Let $f : [a,b]\times[c,d] \to \mathbb{R}$ be a continuous function. Then*

$$\int_a^b \left(\int_c^d f(x,y)\mathrm{d}y\right) \mathrm{d}x = \int_c^d \left(\int_a^b f(x,y)\,\mathrm{d}x\right) \mathrm{d}y$$

Proof. Since f is continuous on $[a,b]\times[c,d]$ Theorem 2.20 implies that $\int_c^d f(x,y)\mathrm{d}y$ and $\int_a^b f(x,y)\,\mathrm{d}x$ are continuous on their respective domains, and therefore Riemann integrable. Consider

$$F(t) = \int_a^t \left(\int_c^d f(x,y)\mathrm{d}y\right) \mathrm{d}x - \int_c^d \left(\int_a^t f(x,y)\,\mathrm{d}x\right) \mathrm{d}y.$$

By the FTC (Theorem 1.30), we know that F is continuous, with $F(a) = 0$. Also the first integral is differentiable with

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_a^t \left(\int_c^d f(x,y)\mathrm{d}y\right) \mathrm{d}x = \int_c^d f(t,y)\mathrm{d}y.$$

We know that

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_a^t f(x,y)\,\mathrm{d}x = f(t,y).$$

We would know like to differentiate the second integral in F, namely

$$-\int_c^d \left(\int_a^t f(x,y)\,\mathrm{d}x\right) \mathrm{d}y$$

by differentiating inside the first integral. For that Theorem 2.21 requires that

$$\left(\int_a^t f(x,y)\,\mathrm{d}x\right)$$

be continuous in $[a,b]\times[c,d]$ as a function of t and y. Theorem 2.20 proves that it is continuous as a function of y and we actually know that it is differentiable as a function of t. However, continuity in each of the variables separately does not ensure that the function is continuous on $[a,b]\times[c,d]$. However, one can modify the proof of the Theorem 2.20 to show continuity in $[a,b]\times[c,d]$. (This is left as an exercise.) Then we are allowed to differentiate inside the integral and we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_c^d\left(\int_a^t f(x,y)\,\mathrm{d}x\right)\mathrm{d}y=\int_c^d\frac{\partial}{\partial t}\left(\int_a^t f(x,y)\,\mathrm{d}x\right)\mathrm{d}y=\int_c^d f(t,y)\,\mathrm{d}y.$$

Therefore

$$F'(t)=\int_c^d f(t,y)\,\mathrm{d}y-\int_c^d f(t,y)\,\mathrm{d}y=0,$$

Since F is continuous on $[a,b]$, $F(a)=0$ and $F'(t)=0$ we find $F(b)=0$. This implies the result. □

Remark 2.23. *The continuity requirement is necessary in the previous Theorem. The following is a counterexamples to Fubini's theorem when continuity fails at just a point. Let*

$$f(x,y)=\frac{x^2-y^2}{(x^2+y^2)^2}.$$

Notice that f is not continuous at the origin. We have

$$\int_0^1\frac{x^2-y^2}{(x^2+y^2)^2}\mathrm{d}y=\left.\frac{y}{x^2+y^2}\right|_{y=0}^{y=1}=\frac{1}{1+x^2},$$

and

$$\int_0^1\left(\int_0^1\frac{x^2-y^2}{(x^2+y^2)^2}\mathrm{d}y\right)\mathrm{d}x=\int_0^1\frac{1}{1+x^2}\mathrm{d}x=\frac{\pi}{4}.$$

In the opposite direction we get $\frac{-\pi}{4}$ by symmetry. The key here is that the function is not in L^1, i.e. $|f|$ is not integrable.

$$\int_{I\times I}\left|\frac{x^2-y^2}{(x^2+y^2)^2}\right|\mathrm{d}A\geq\int_0^1\left(\int_0^x\frac{x^2-y^2}{(x^2+y^2)^2}\mathrm{d}y\right)\mathrm{d}x=\int_0^1\left.\frac{y}{x^2+y^2}\right|_{y=0}^{y=x}=\int_0^1\frac{1}{2x}\mathrm{d}x=\infty.$$

Differentiation revisited.

We will use the notation $C^k(a,b)$ to denote functions that are continuously differentiable on (a,b), and $C^\infty(a,b)$ for functions that are infinitely differentiable on (a,b).

We have seen examples of sequences (f_n) that are differentiable, with (f_n) converging uniformly to f but for which f_n' does not converge to f'. In fact it is easy to construct examples of C^1 functions that converge uniformly for which f' does not exist. Consider

$$f_n(x)=\left(x^2+1/n\right)^{1/2}.$$

They are clearly C^1 as the x^2+1/n never vanishes for fixed n. (f_n) converges uniformly to $f(x)=|x|$, which is not smooth at the origin. To see this notice that if

$$A:=\left(x^2+1/n\right)^{1/2}-|x|$$

then

$$A\leq\left((x+1/\sqrt{n})^2\right)^{1/2}-|x|\leq\frac{1}{\sqrt{n}},$$

and the uniform convergence follows.

The following result will prove rather useful.

Theorem 2.24. *Let (f_n) be a sequence of C^1 functions on $[a, b]$ (understood as a one-sided derivative). Assume $f_n \to f$ in the pointwise sense and that f_n' converges uniformly to g. Then f is C^1 and $g = f'$ or $f_n' \to f'$.*

Proof. Since $f_n' \rightrightarrows g$, Theorem 2.16 yields

$$\int_a^x g(y)\mathrm{d}y = \int_a^x \lim_{n\to\infty} f_n' = \lim_{n\to\infty} \int_a^x f_n',$$

which by the FTC yields

$$\int_a^x g(y)\mathrm{d}y = \lim_{n\to\infty} [f_n(x) - f_n(a)] = f(x) - f(a).$$

Notice that since g is continuous this means that f is continuous. While the Theorem does not assume that g is continuous, that is a consequence of the uniform convergence of f_n' to g, since f_n are C^1. Now, the FTC implies that $\int_a^x g$ is differentiable with derivative g. Since

$$\int_a^x g = f(x) - f(a)$$

we obtain that f is differentiable and $g = f'$. □

2.2 Series of functions

In this section we consider series of functions, i.e., we study

$$\sum_{k=1}^{\infty} f_n(x),$$

with $f_n : \Omega \to \mathbb{R}$. We begin by establishing the notion of pointwise convergence and uniform convergence for a series.

Definition 2.25. *Let (f_k) be a sequence of functions $f_k : \Omega \to \mathbb{R}$. Let (S_n) be the sequence of partial sums, with $S_n : \Omega \to \mathbb{R}$ defined by*

$$S_n(x) = \sum_{k=1}^{n} f_k(x).$$

Then the series

$$\sum_{k=1}^{\infty} f_k(x)$$

converges pointwise to $S : \Omega \to \mathbb{R}$ in Ω if $S_n \to S$ pointwise in Ω and it converges uniformly to S in Ω if $S_n \rightrightarrows S$ uniformly on Ω.

Theorem 2.26. *Let (f_k), with $f_k : [a, b] \to \mathbb{R}$, be a sequence of integrable functions. Assume that $S_n = \sum_{k=1}^{n} f_k$ converges uniformly. Then $\sum_{k=1}^{\infty} f_k$ is Riemann integrable and*

$$\int \sum_{k=1}^{\infty} f_k = \sum_{k=1}^{\infty} \int f_k.$$

Proof. S_n is a finite sum of integrable functions and therefore integrable (by additivity). Since S_n converges uniformly Theorem 2.16 implies that S is integrable and moreover

$$\lim_{n\to\infty} \int S_n = \int \lim_{n\to\infty} S_n.$$

Since $\int S_n = \sum_{k=1}^{n} \int f_k$ and $\lim_{n\to\infty} S_n = \sum_{k=1}^{\infty} f_k$ we obtain the result. □

Theorem 2.27. *Let (f_k), with $f_k : [a, b] \to \mathbb{R}$, be a sequence of C^1 functions such that $S_n = \sum_{k=1}^n f_k$ converges pointwise. Assume that $\sum_{k=1}^n f_k'$ converges uniformly. Then*

$$\left(\sum_{k=1}^{\infty} f_k(x)\right)' = \sum_{k=1}^{\infty} f_k'(x),$$

that is, the series is differentiable and can be differentiated term-by-term.

Proof. The proof is a simple consequence of Theorem 2.24.This results says (changing the notation) that if S_n is C^1, $S_n \to S$, $S_n' \rightrightarrows g$ then $S \in C^1$ and $S' = g$ (or $S_n' \rightrightarrows S'$). If we define $S_n = \sum_{k=1}^n f_k$ then, it is C^1, since each f_k is C^1; it converges pointwise to $S = \sum_{k=1}^{\infty} f_k$ and finally S' converges uniformly, to g say. Then S is C^1 and $S_n' \rightrightarrows S'$. This means

$$\lim_{n\to\infty} S_n' = S' = \left(\sum_{k=1}^{\infty} f_k(x)\right)'$$

but since $S_n' = (\sum_{k=1}^n f_k)' = \sum_{k=1}^n f_k'$ we obtain the result, namely

$$\sum_{k=1}^{\infty} f_k'(x) = \left(\sum_{k=1}^{\infty} f_k(x)\right)'.$$

□

Theorem 2.28 (The Weierstrass M-test)**.** *Let (f_k) be a sequence of functions $f_k : \Omega \to \mathbb{R}$, and assume that for every k there exists $M_k > 0$ such that $|f_k(x)| \leq M_k$ for every $x \in \Omega$ and $\sum_{k=1}^{\infty} M_k < \infty$. Then*

$$\sum_{k=1}^{\infty} f_k$$

converges uniformly on Ω.

Proof. Notice that it suffices to show that $S_n := \sum_{k=1}^n f_k(x)$ is uniformly Cauchy. Now since $\sum_{k=1}^{\infty} M_k < \infty$, given $\varepsilon > 0$ there exists N such that

$$\sum_{k=m+1}^{n} M_k < \varepsilon \qquad \text{for all } m, n > N.$$

Now

$$|S_n(x) - S_m(x)| = \left|\sum_{k=1}^{n} f_k(x) - \sum_{k=1}^{m} f_k(x)\right| = \left|\sum_{k=m+1}^{n} f_k(x)\right| \leq \sum_{k=m+1}^{n} |f_k| \leq \sum_{k=m+1}^{n} M_k \leq \varepsilon,$$

for every x. Therefore S_n is uniformly Cauchy and the proof is complete. □

2.3 A continuous, nowhere differentiable function

In 1872 Weierstrass showed that there exist continuous functions that are nowhere differentiable. Standard examples are constructed using Fourier Series. For example

$$f(x) = \sum_{k=0}^{\infty} a^k \cos(2\pi b^k x)$$

for any $0 < a < 1 < b$ with $ab > 1$.

We will construct an example based of the sawtooth function. Consider

$$\phi(x) = \begin{cases} x - \lfloor x \rfloor & x \leq \lfloor x \rfloor + \frac{1}{2} \\ 1 - x + \lfloor x \rfloor & x > \lfloor x \rfloor + \frac{1}{2}. \end{cases}$$

The function ϕ is equal to the distance function from x to $\mathbb{Z}$.

We define, for $n = 0, 1, \dots$

$$f_n(x) = \frac{1}{4^n}\phi(4^n x).$$

We will show that $f(x) = \sum_{n=0}^{\infty} f_n$ is continuous but nowhere differentiable. Notice that

$$0 \leq f_n \leq \frac{1}{4^n}\phi \leq \frac{1}{2}\frac{1}{4^n},$$

and that by the Weierstrass M-test we have the uniform convergence of the series. Since each f_n is continuous, and the convergence is uniform we have that f is C^0.

Given $x \in \mathbb{R}$ we will choose the sign of $h_n = \pm\frac{1}{4^{n+1}}$ in such a way that the points $4^n x$ and $4^n(x + h_n)$ both belong to the same interval of length 1/2, $[\frac{k}{2}, \frac{k+1}{2}]$ for some $k \in \mathbb{Z}$. We make this choice of sign for h_n because on each of these intervals $[\frac{k}{2}, \frac{k+1}{2}]$, the function ϕ has constant slope $+1$ or -1.

Consider the incremental quotient

$$\frac{f_n(x + h_n) - f_n(x)}{h_n} = \frac{\phi(4^n x + 4^n h_n) - \phi(4^n x)}{4^n h_n} = \pm 1.$$

Moreover, if $m < n$ the graph off_m also has slope ± 1 on the interval in which x and $x + h_n$ belong to. Therefore

$$\epsilon_m := \frac{f_m(x + h_n) - f_n(x)}{h_n} = \frac{\phi(4^m x + 4^m h_n) - \phi(4^m x)}{4^m h_n} = \pm 1.$$

However for $m \geq n + 1$ we have (since $4^m x + 4^m h_n - 4^m x = +4^m h_n = \pm 4^{m-n-1} \in \mathbb{N}$)

$$f_m(x + h_n) - f_n(x) = \phi(4^m x + 4^m h_n) - \phi(4^m x) = 0.$$

Therefore

$$A_n := \frac{f(x + h_n) - f(x)}{h_n} = \sum_{m=0}^{n} \frac{f_m(x + h_n) - f_m(x)}{h_n} = \sum_{m=0}^{n} \epsilon_m.$$

Therefore A_n is an even integer if n is odd and an odd integer if n is even. Hence there is no limit as n goes to infinity. Since h_n goes to zero that proves that f is not differentiable.

2.4 Space filling curves

In this section we will show the existence of surjective, continuous maps $\gamma : [0,1] \to [0,1] \times [0,1]$ by looking at Hilbert's curve. This is just one example of what is known as space-filling curves.

The curve is constructed as the limit of an iterative construction, similarly to the Devil's staircase.

We start with the unit square, and divide it up into 4 equal squares. On the left of Figure 2.2 the curve in blue is the building block of the first iterate. It connects the centres of the 4 squares using 3 straight segments. Any choice would yield a similar curve, but once we have chosen one we will use the same order (in this case starting in the bottom-left square and moving clockwise) in the rest of the iterates. We extend the curve with the two green segments to join it to the boundary. The total length of the curve is 2. We construct $\gamma_1 : [0,1] \to [0,1] \times [0,1]$ running along the described curve at uniform speed. That means $[0, 1/4]$ is mapped to the section of the curve in the bottom left square, $[1/4, 1/2]$ to the top left, $[1/2, 3/4]$ to the top tight and $[3/4, 1]$ to the bottom right.

In order to construct the second iterate, we want to bisect every square in our previous grid. In this case we will have 16 squares. In each block of 4 we copy a scaled down version of the building block in

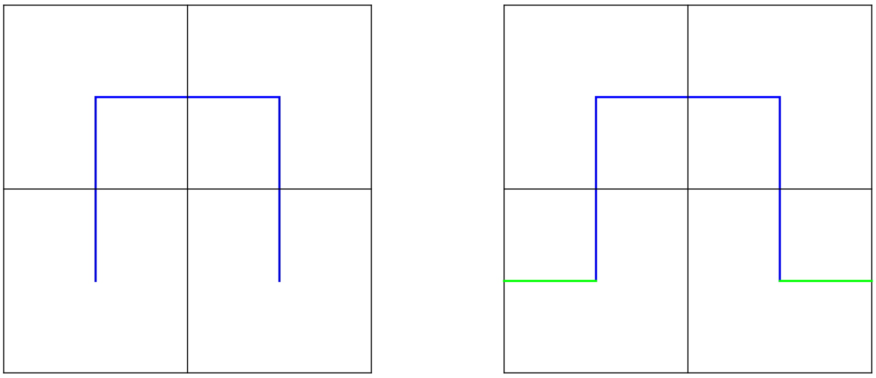

Figure 2.2: First iterate in the construction of the Hilbert curve

the previous iteration (see left-hand side of Figure 2.2), but changing the orientation as indicated in the left-hand side of Figure 2.3. To complete the curve we join the pieces using the 3 segments in red and connect the curve to the boundary using the two segments in green (see RHS of Figure (2.3)).

Note that the length of this curve is double what it was in the previous iteration. We construct $\gamma_2 : [0,1] \to [0,1] \times [0,1]$ by running the curve at uniform speed. Notice that the curve moves along the 16 squares, and in each the length of the curve is 1/4.

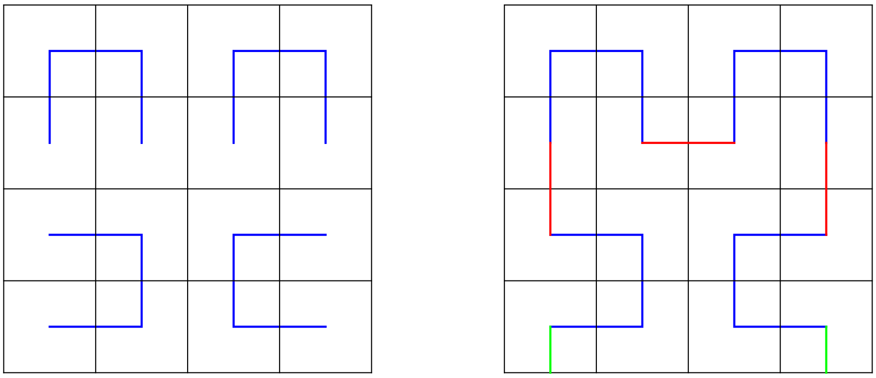

Figure 2.3: First iterate in the construction of the Hilbert curve

To construct the third iterate we proceed as before. Now we use the curve from the previous iteration (without the green segments), and place 4 scaled down copies in the new grid, performing the same rotations used in the previous iterations. See the left hand side of Figure 2.4. As before we join the 4 curves with the segments in red and link the curve to the boundary with the segments in green.

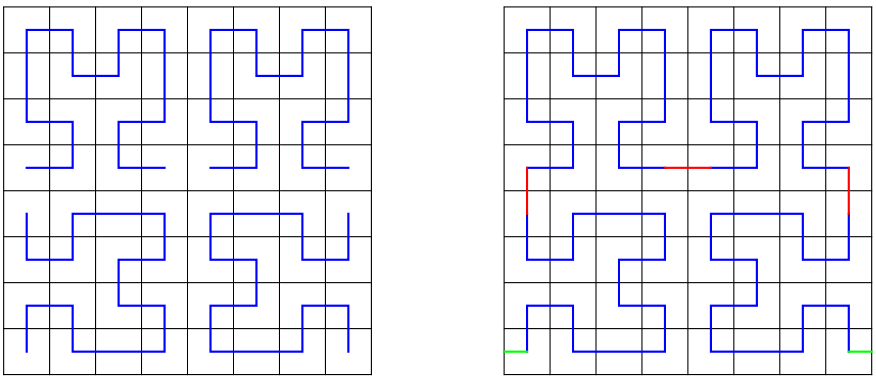

Figure 2.4: First iterate in the construction of the Hilbert curve

In this iterate there are 64 squares, and the length of the curve has doubled again to 8.

In this fashion we can construct $\gamma_n : [0,1] \to [0,1] \times [0,1]$ which is continuous and runs through the grid with 2^{2n} squares, in particular runs through the centres of all those squares. If we divide $[0,1]$ in 4^k intervals, in each of them we run through one square.

If we show that γ is a continuous curve we obtain the result. Since the image of a compact set by a continuous map is compact the range of γ must be the unit square (notice that γ_n runs through the centres of all the squares, and therefore we can approximate any point in $[0,1] \times [0,1]$ with points in the image of γ_n).

In order to prove that γ is continuous notice that if we consider an interval $I \subset [0,1]$ of length $|I| < \frac{1}{4^k}$ then $\gamma_n(I)$ is contained in at most two squares of sides $1/2^k$ provided $n \geq k$.

Therefore the Euclidean distance between $\gamma_n(s)$ and $\gamma_n(t)$ is control by

$$|\gamma_n(s) - \gamma_n(t)| \leq \frac{\sqrt{5}}{2^k},$$

the furthest possible distance between two adjacent squares of side $\frac{1}{2^k}$.

Given t, s we find k such that

$$\frac{1}{4^{k+1}} < |t - s| \leq \frac{1}{4^k}.$$

Now for $n \geq k$

$$|\gamma_n(t) - \gamma_n(s)| \leq \frac{\sqrt{5}}{2^k} = \frac{1\sqrt{5}}{\sqrt{4^{k+1}}} \leq 2\sqrt{5}|t - s|^{1/2}.$$

Therefore, taking limits as n goes to infinity we find

$$|\gamma(t) - \gamma(s)| \leq c|t - s|^{1/2},$$

with $c = \sqrt{6}$. This proves that γ is continuous.

We remark that the curve cannot be injective. Indeed, if it was a simple curve (i.e. non-self-intersecting), the Jordan curve Theorem would imply that the curve divides the plane into an interior and an exterior region, so that any curve joining a point from the interior with one from the exterior would cross the curve. To make this a bit more precise we include a formal definition and the main theorem.

Definition 2.29. *A Jordan curve C in $\mathbb{R}^2$ is the image of an injective, continuous map of a circle, $\varphi : \mathbb{S}^1 \to \mathbb{R}^2$.*

Theorem 2.30. *Let C be a Jordan curve in $\mathbb{R}^2$. Then the complement $\mathbb{R}^2 \backslash C$ consists of exactly two connected components. One of the components is bounded (the interior) and the other is unbounded (the exterior) and the curve is the boundary of each component.*

Notice that the curve cannot be differentiable or have a notion of tangent that makes it possible to define the left-hand side and the right-hand side, which would correpond the two connected components of the theorem.

2.5 Absolute Continuity

Definition 2.31. *Let I be in an interval in $\mathbb{R}$. A function $f : I \to \mathbb{R}$ is increasing (strictly increasing) if $f(x) \leq f(y)$ ($f(x) < f(y)$) whenever $x, y \in I$ and $x < y$. Similarly $f : I \to \mathbb{R}$ is decreasing (strictly decreasing) if $f(x) \geq f(y)$ ($f(x) > f(y)$) whenever $x, y \in I$ and $x < y$.*

Theorem 2.32. *Let $f : [a,b] \to \mathbb{R}$ be an increasing function. Then f is differentiable almost everywhere. Moreover*

$$\int_a^b f'(x)\mathrm{d}x \leq f(b) - f(a).$$

Notice that an analogous result (reversing the inequality) is true for decreasing functions.

Definition 2.33. *Given $f : [a, b] \to \mathbb{R}$ the total variation of f over $[a, b]$ is*

$$Vf := \sup\{\sum_{i=1}^{n} |f(x_i) - f(x_{i-1})|\},$$

where the supremum is taken over all possible partitions $a = x_0 < x_1 < \cdots < x_{n-1} < x_n = b$ of $[a, b]$. A function f is of bounded variation if Vf is finite.

Theorem 2.34. *A function $f : [a, b] \to \mathbb{R}$ is of bounded variation if and only if f is the difference of two monotone functions on $[a, b]$.*

Proof. We define

$$T_f(x) := \sup\{\sum_{j=1}^{n} |f(x_j) - f(x_{j-1})|\},$$

where the supremum is taken over all $n \in \mathbb{N}$ and all partitions $a = x_0 < x_1 < \cdots < x_n = x$ of the interval $[a, x]$. Notice that if we add a point to any such partition the sum in the definition is made bigger, and therefore we have

$$T_f(x) \leq T_f(y) \qquad x < y,$$

i.e. T_f is increasing. We claim that $T_f + f$ and $T_f - f$ are increasing. This will prove the result as

$$f = \frac{1}{2}[T_f + f] - \frac{1}{2}[T_f - f].$$

To prove the claim, let $x < y$ and $\varepsilon > 0$. Choose a partition $a = x_0 < x_1 < \ldots < x_n = x$ such that

$$\sum_{j=1}^{n} |f(x_j) - f(x_{j-1})| \geq T_f(x) - \varepsilon.$$

Then

$$\sum_{j=1}^{n} |f(x_j) - f(x_{j-1})| + |f(y) - f(x)|$$

is an approximation for $T_f(y)$, and it is less than or equal to $T_f(y)$. Now, since $x < y$, with the partition above for $[a, x]$

$$T_f(y) + f(y) \geq \sum_{j=1}^{n} |f(x_j) - f(x_{j-1})| + |f(y) - f(x)| + f(y)$$

$$\geq \sum_{j=1}^{n} |f(x_j) - f(x_{j-1})| + \underbrace{|f(y) - f(x)| + f(y) - f(x)}_{\geq 0} + f(x) \geq T_f(x) - \varepsilon + f(x).$$

Since $\varepsilon > 0$ is arbitrary we obtain the result for $T_f + f$. Notice that the results is the same for $T_f - f$. Namely

$$T_f(y) - f(y) \geq \sum_{j=1}^{n} |f(x_j) - f(x_{j-1})| + |f(y) - f(x)| - f(y)$$

$$\geq \sum_{j=1}^{n} |f(x_j) - f(x_{j-1})| + \underbrace{|f(y) - f(x)| - f(y) + f(x)}_{\geq 0} - f(x) \geq T_f(x) - \varepsilon - f(x).$$

□

As a consequence, if f is of bounded variation on $[a, b]$ then $f'(x)$ exists for almost every $x \in [a, b]$.

Definition 2.35. *A function $f : [a,b] \to \mathbb{R}$ is absolutely continuous if for every $\varepsilon > 0$ there exists $\delta > 0$ such that*

$$\sum_{i=1}^{n} |f(b_i) - f(a_i)| < \varepsilon$$

for every n and every disjoint collection of intervals $(a_1, b_1), \dots, (a_n, b_n)$ with

$$\sum_{i=1}^{n} b_i - a_i < \delta.$$

Notice that since we can take $n = 1$ in the definition above, functions that are absolutely continuous are continuous, and that since δ cannot depend on x they are also uniformly continuous. However, if we consider

$$f(x) = x \sin \frac{1}{x}$$

on $[-1, 1]$, the function if continuous (and therefore uniformly continuous) but it is not absolutely continuous.

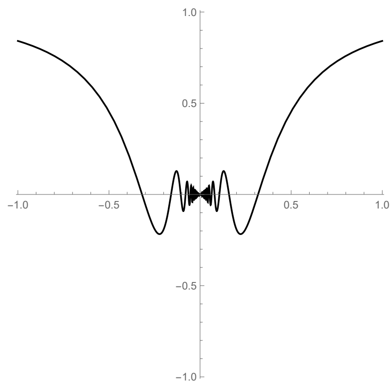

Figure 2.5: Graph of $x \sin(1/x)$.

This can be seen by showing that it is not of bounded variation, by carefully choosing partitions where $\sin(1/x)$ equals $+1$ and -1 at the endpoints.

Theorem 2.36. *Let $f : [a,b] \to \mathbb{R}$ be continuous and nondecreasing. The following three statements are equivalent:*

1. *f is absolutely continuous on $[a,b]$.*
2. *f maps sets of measure 0 to sets of measure 0.*
3. *f is differentiable almost everywhere on $[a,b]$, $f' \in L^1$ (i.e. f' is integrable) and*

$$\int_a^x f'(t) \mathrm{d}t = f(x) - f(a).$$

Chapter 3

Complex Analysis

This part of the course is an introduction to complex analysis. The main topics will be complex differentiability, power series and contour integrals. Basic notions and properties for complex numbers were introduction in Year I and we only provide a quick review here.

3.1 Review of basic facts about $\mathbb{C}$

The field of complex numbers is given by

$$\mathbb{C} = \{z = x + \mathrm{i}y, \quad x, y \in \mathbb{R}\},$$

with $\mathrm{i}^2 = -1$. For $z = x + \mathrm{i}y$ as above we say that x is the real part of z, denoted by $x = \mathbf{Re}\, z$ and that y is the imaginary part of z, denoted by $y = \mathbf{Im}\, z$. By $|z|$ we denote the modulus (or norm) of z, given by $\sqrt{x^2 + y^2}$. We denote by $\bar{z}$ the complex conjugate of z. That is, if $z = x + \mathrm{i}y$ then $\bar{z} = x - \mathrm{i}y$. It is easy to see that

1. $\bar{\bar{z}} = z$,
2. $\overline{z + w} = \bar{z} + \bar{w}$,
3. $\overline{zw} = \bar{z}\bar{w}$,
4. $|z|^2 = z\bar{z}$ and $|\bar{z}| = |z|$.

Notice that we can identify $\mathbb{C}$ with $\mathbb{R}^2$, simply by identifying $z = x + \mathrm{i}y$ with (x, y). In this way $|z|$ corresponds to the Euclidean norm in $\mathbb{R}^2$. We also derive the notions of convergence, open and closed for $\mathbb{C}$ from this analogy.

Definition 3.1. *We say that $(z_n)_{n=1}^{\infty} \subset \mathbb{C}$ converges to z if and only if $|z_n - z|$ tends to zero as n goes to infinity. That is, if for every $\varepsilon > 0$ there exists $N > 0$ such that $|z_n - z| < \varepsilon$ for all $n > N$.*

Definition 3.2. *We say that $\Omega \subset \mathbb{C}$ is open if and only for every $x \in \Omega$ there exits $r > 0$ such that $B_r(x) = \{z \in \mathbb{C} |\, |z - x| < r\} \subset \Omega$. We say that Ω is closed if and only if Ω^c is open.*

Definition 3.3. *A set $K \subset \mathbb{C}$ is sequentially compact if and only if for every sequence $(x_j)_{j \in \mathbb{N}} \subset K$ has a convergent subsequence $(x_{j(l)})_{l \in \mathbb{N}}$ whose limit is in K.*

Now, maps in $f : \Omega \subset \mathbb{C} \to \mathbb{C}$, are given by a pair of real-functions $f(z) = u(z) + \mathrm{i}v(z)$, the real part u of f and the imaginary part v of f. We can think of those two functions as functions of z or as functions in $\mathbb{R}^2$ of x and y, the real and imaginary part of z. This means we can also think of f as a function from $\Omega \subset \mathbb{R}^2$ to $\mathbb{R}^2$.

Definition 3.4. *Given $f : \Omega \subset \mathbb{C} \to \mathbb{C}$ we say that it is continuous at $z_0 \in \Omega$ if and only if for every $\varepsilon > 0$ there exists δ such that $|z - z_0| < \delta$, with $z \in \Omega$ implies that $|f(z) - f(z_0)| < \varepsilon$.*

Notice that the notion of continuity coincides with the one defined in MA259 for maps from $\mathbb{R}^2$ to $\mathbb{R}^2$. We will now consider the notion of differentiability, where the two notions differ very significantly.

Recall (from MA259) that a function $f : \mathbb{R}^n \to \mathbb{R}^k$ is differentiable at a point p if and only if there exists a linear map $Df(p) \in L(\mathbb{R}^n; \mathbb{R}^k)$ such that

$$\lim_{h\to 0} \frac{|f(p+h) - f(p) - Df(p)h|}{|h|} = 0, \tag{3.1}$$

where the norms above refer to norms in $\mathbb{R}^k$ in the numerator and in $\mathbb{R}^n$ in the denominator. The reason for introducing that definition arose from the fact that when $k > 1$ we have no notion of division for the quantity we would like to study

$$\lim_{h\to 0} \frac{f(p+h) - f(p)}{h}$$

as division by $h \in \mathbb{R}^n, n > 1$ is not well defined. However, in $\mathbb{C}$ we do have a notion of multiplication and therefore we can use that quotient to define differentiability.

Definition 3.5. *Let $\Omega \subset \mathbb{C}$ be an open set and $z \in \Omega$. We say that f is complex differentiable at z if and only if the limit*

$$\lim_{h\to 0} \frac{f(z+h) - f(z)}{h} \tag{3.2}$$

exists. We denote the limit by $f'(z)$.

In contrast to what happened in the real valued case, where the derivative was a linear map from $\mathbb{R}^n$ to $\mathbb{R}^k$, which in our case would mean from $\mathbb{R}^2$ to $\mathbb{R}^2$, corresponding to a 2 by 2 matrix, in the complex case we obtain a complex number. Before studying how to reconcile this difference, we look at the consequences of the definition for the real and imaginary part of f. Let's write $h = \Delta x + \mathrm{i}\Delta y$, and $f(z) = u(z) + \mathrm{i}v(z)$, which we can also think of as $f(x,y) = u(x,y) + \mathrm{i}v(x,y)$. Then the quotient in the definition of complex derivative can be rewritten as

$$\frac{f(z+h) - f(z)}{h} = \frac{u(x+\Delta x, y+\Delta y) - u(x,y) + \mathrm{i}[v(x+\Delta x, y+\Delta y) - v(x,y)]}{\Delta x + \mathrm{i}\Delta y}.$$

We could consider multiple ways of sending $\Delta x + \mathrm{i}\Delta y$ to zero, obtaining the same answer if the limit exists. We will consider the two obvious options, sending Δx first to zero followed by Δy, and the reverse, Δy first followed by Δx. We find

$$\begin{aligned}
&\lim_{\Delta y\to 0}\lim_{\Delta x\to 0} \frac{u(x+\Delta x, y+\Delta y) - u(x,y) + \mathrm{i}[v(x+\Delta x, y+\Delta y) - v(x,y)]}{\Delta x + \mathrm{i}\Delta y} \\
&\quad= \lim_{\Delta y\to 0} \frac{u(x, y+\Delta y) - u(x,y) + \mathrm{i}[v(x, y+\Delta y) - v(x,y)]}{\mathrm{i}\Delta y} \\
&\quad= \frac{1}{\mathrm{i}} \lim_{\Delta y\to 0} \frac{u(x, y+\Delta y) - u(x,y)}{\Delta y} + \mathrm{i}\frac{v(x, y+\Delta y) - v(x,y)}{\Delta y} \\
&\quad= \frac{1}{\mathrm{i}}\left[\frac{\partial u}{\partial y}(x,y) + \mathrm{i}\frac{\partial v}{\partial y}(x,y)\right] = v_y(x,y) - \mathrm{i}u_y(x,y),
\end{aligned}$$

while

$$\begin{aligned}
&\lim_{\Delta x\to 0}\lim_{\Delta y\to 0} \frac{u(x+\Delta x, y+\Delta y) - u(x,y) + \mathrm{i}[v(x+\Delta x, y+\Delta y) - v(x,y)]}{\Delta x + \mathrm{i}\Delta y} \\
&\quad= \lim_{\Delta x\to 0} \frac{u(x+\Delta x, y) - u(x,y) + \mathrm{i}[v(x+\Delta x, y) - v(x,y)]}{\Delta x} \\
&\quad= \lim_{\Delta x\to 0} \frac{u(x+\Delta x, y) - u(x,y)}{\Delta x} + \mathrm{i}\frac{v(x+\Delta x, y) - v(x,y)}{\Delta x}
\end{aligned}$$

$$= \left[\frac{\partial u}{\partial x}(x,y) + \mathrm{i}\frac{\partial v}{\partial x}(x,y)\right] = u_x(x,y) + \mathrm{i}v_x(x,y).$$

This immediately means that at the very least we need to demand some relationships between the partial derivatives of u and v to hold in order to have a complex differential. Namely

$$u_x = v_y \qquad u_y = -v_x \tag{3.3}$$

This equations are known as the Cauchy–Riemann equations. These are clearly necessary conditions, but at this point in no way guarantee that a complex differential would exists if satisfied.

By considering two simple examples, it is easy to see that the notion of complex differential is highly restrictive as functions that are obviously smooth when considered as a map from $\mathbb{R}^2$ to $\mathbb{R}^2$ are not actually complex differentiable. First we consider $f(z) = z$. Notice that $f'(z)$ exists and equals 1. Indeed

$$f'(z) = \lim_{h\to 0}\frac{z+h-z}{h} = 1.$$

However if we consider $g(z) = \bar{z}$, we obtain a function that is not complex differentiable. We have

$$\lim_{h\to 0}\frac{g(z+h)-g(z)}{h} = \lim_{h\to 0}\frac{\bar{z}+\bar{h}-\bar{z}}{h} = \lim_{h\to 0}\frac{\bar{h}}{h},$$

a limit that does not exist. (Consider for example the limits obtained by taking h along the real or the imaginary axis.) The function g does not satisfy the Cauchy–Riemann equations. We have $g(z) = x - \mathrm{i}y$, and therefore

$$u_x = 1, \quad v_y = -1, \quad u_y = 0, \quad v_x = 0.$$

When considering g as a function from $\mathbb{R}^2$ to $\mathbb{R}^2$ we have $g(x,y) = (x,-y)$ we clearly have a differentiable function, as all components are smooth functions. (The existence of continuous partial derivatives suffices to obtain differentiability, as seen in MA259.)

Definition 3.6. *We say that $f : \Omega \to \mathbb{C}$ is analytic (or holomorphic) in a neighbourhood U of z if it is complex differentiable everywhere in U. We say that f is entire if it is analytic in the whole of $\mathbb{C}$.*

A function can be differentiable at one point, but not necessarily analytic. Consider as an example the function $f(z) = |z|^2$. We will show that the function is complex differentiable at 0, but that it is not analytic, as it is not complex differentiable outside the origin. Notice that $f(z) = x^2 + y^2$, and $u = x^2 + y^2$ and $v = 0$. When computing the Cauchy–Riemann equations we find

$$u_x = 2x \qquad u_y = 2y, \qquad v_x = v_y = 0.$$

The Cauchy–Riemann equations mean $2x = 0$ and $2y = 0$, which is only satisfied at the origin. Now, to check that f is complex differentiable at the origin

$$\left.\frac{|z+h|^2 - |z|^2}{h}\right|_{z=0} = \frac{|h|^2}{h} = \bar{h} \xrightarrow[h\to 0]{} 0,$$

proving that f is complex differentiable at the origin with derivative 0.

We will now revisit the Cauchy–Riemann equations and connect complex differentiability with the dependence of the function on $\bar{z}$. Consider $f(z)$ as given by $u(x,y) + \mathrm{i}v(x,y)$. Using the fact that $x = \frac{z+\bar{z}}{2}$ and $y = \frac{z-\bar{z}}{2\mathrm{i}}$ we can rewrite the function back in terms of z and $\bar{z}$. Now, we could consider the derivative of f with respect to $\bar{z}$. Applying the chain rule we would obtain

$$\frac{\partial u}{\partial \bar{z}} = u_x\frac{1}{2} - u_y\frac{1}{2\mathrm{i}} \qquad\qquad \frac{\partial v}{\partial \bar{z}} = v_x\frac{1}{2} - v_y\frac{1}{2\mathrm{i}}$$

Therefore

$$\frac{\partial f}{\partial \bar{z}} = u_{\bar{z}} + \mathrm{i}v_{\bar{z}} = u_x\frac{1}{2} - u_y\frac{1}{2\mathrm{i}} + \mathrm{i}\left[v_x\frac{1}{2} - v_y\frac{1}{2\mathrm{i}}\right],$$

which we can simplify to

$$\frac{\partial f}{\partial \bar{z}} = \frac{1}{2}\left[u_x - v_y\right] + \mathrm{i}\frac{1}{2}\left[v_x + u_y\right].$$

Notice that if the function is complex differentiable, it satisfies the Cauchy–Riemann equations and therefore the expression above is identically zero. In this sense we say that if a function is complex differentiable, then

$$\frac{\partial f}{\partial \bar{z}} = 0.$$

This illustrates why $f(z) = \bar{z}$ or $g(z) = |z|^2 = z\bar{z}$ were not complex differentiable.

Now that we know that the Cauchy–Riemann equations need to be satisfied for a function to be complex differentiable we can identify the complex plane with a subspace of 2×2 matrices. This identification will allow us to connect directly complex differentiability with the standard notion of differentiability from MA259. We have already identified $a + \mathrm{i}b$ with the point in $\mathbb{R}^2$ given by (a, b). We can also identify it with the matrix

$$\begin{pmatrix} a & -b \\ b & a \end{pmatrix}.$$

Note that which factor of b contains a minus sign is just a convention. Notice that the determinant of that matrix equals $|a + \mathrm{i}b|^2$, and that therefore the matrix is invertible unless $a + \mathrm{i}b = 0$. This identification preserves the basic operations we have for complex numbers, for example summation and multiplication. That is it is possible to perform the operation $(a + \mathrm{i}b) + (c + \mathrm{i}d)$ as complex numbers or as the sum of the two corresponding matrices, with the results agreeing (modulo the identification). For the product we have

$$(a + \mathrm{i}b)(c + \mathrm{i}d) = (ac - bd) + \mathrm{i}(bc + ad)$$

and

$$\begin{pmatrix} a & -b \\ b & a \end{pmatrix}\begin{pmatrix} c & -d \\ d & c \end{pmatrix} = \begin{pmatrix} ac - bd & -(bc + ad) \\ bc + ad & ac - bd \end{pmatrix},$$

proving the result. Sometimes it is useful to consider a hybrid of both identification, the one as a matrix, and the one as a point (or vector) in $\mathbb{R}^2$. For example, for the product of two complex numbers that we have just considered, we could identify it with

$$\begin{pmatrix} a & -b \\ b & a \end{pmatrix}\begin{pmatrix} c \\ d \end{pmatrix}.$$

The answer is the vector

$$\begin{pmatrix} ac - bd \\ bc + ad \end{pmatrix},$$

which corresponds to the right complex number $(ac-bd)+\mathrm{i}(bc+ad)$ and to the matrix $\begin{pmatrix} ac - bd & -(bc + ad) \\ bc + ad & ac - bd \end{pmatrix}$.

We are now ready to connect complex differentiation with Cauchy–Riemann and differentiation for functions in $\mathbb{R}^2$.

Theorem 3.7. *Let $f : \Omega \subset \mathbb{C} \to \mathbb{C}$ with Ω open. f is complex differentiable at $z = a + \mathrm{i}b \in \Omega$ if and only if f, when considered as map from $\Omega \subset \mathbb{R}^2$ to $\mathbb{R}^2$ has a differential at the point (a, b) that satisfies the Cauchy–Riemann equation.*

Before we prove this result, we emphasize that some books will replace the right-hand side by asking that the Cauchy–Riemann equations are satisfied and that all partial derivatives are continuous. Notice that this last condition implies the existence of a differential (as proven in MA259).

Proof. Assume that f is complex differentiable at $z = a + \mathrm{i}b$. Then we have

$$\lim_{h\to 0} \frac{f(z+h) - f(z)}{h} = f'(z),$$

which we can rewrite as

$$\lim_{h\to 0} \left| \frac{f(z+h) - f(z) - f'(z)h}{h} \right| = 0. \tag{3.4}$$

In order to prove that f is differentiable as a map in $\mathbb{R}^2$ we need to find a linear map Df that satisfies (3.1), which translates in finding a 2×2 matrix. Notice that (3.4) suggest that $f'(z) \in \mathbb{C}$ should be the map. Indeed if we identify $f'(z)$ with the corresponding matrix, and think of $f'(z)h$ not as a product of two complex numbers but as a matrix acting on the vector h then we have in fact proven that f has a differential. Since we already know that all complex differentiable functions satisfy the Cauchy–Riemann equations we have completed that implication.

For the reverse, assuming that we have a differential, that means that we have a 2×2 matrix which is given by

$$Df((a,b)) = \begin{pmatrix} u_x & u_y \\ v_x & v_y \end{pmatrix}$$

and that satisfies

$$\lim_{h\to 0} \frac{|f((a,b)+h) - f((a,b)) - Df((a,b))h|}{|h|} = 0.$$

Since the Cauchy–Riemann equations are satisfied we know that this matrix does in fact have the form

$$\begin{pmatrix} u_x & -v_x \\ v_x & u_x \end{pmatrix},$$

meaning that we could identify it with a complex number as before. We could therefore identify $Df\,h$ with the product of the complex numbers $f'(z) = u_x + \mathrm{i}v_x$ and h. Identifying (a,b) with z we obtain

$$\lim_{h\to 0} \frac{|f(z+h) - f(z) - f'(z)h|}{|h|} = 0$$

which implies that

$$\lim_{h\to 0} \frac{f(z+h) - f(z)}{h}$$

exists and equals $f'(z)$, completing the proof. □

As a consequence of the above result, since we can connect complex differentials with differentials as maps from $\mathbb{R}^2$ to $\mathbb{R}^2$, the results from MA259 directly yield the following:

Theorem 3.8. *Lef $f, g : \Omega \subset \mathbb{C} \to \mathbb{C}$ be complex differentiable functions. Then (asumming $g \neq 0$ in the third expression) we have that the familiar expressions*

$$(f+g)' = f' + g' \qquad (fg)' = f'g + fg' \qquad \left(\frac{f}{g}\right)' = \frac{f'g - fg'}{g^2} \qquad (f(g))' = f'(g)g'$$

apply to the complex-valued case as well. For the final expression one needs to assume that the composition makes sense, i.e. the range of g is contained in the domain of f.

We conclude this section by proving that $f(z) = z^n$ is complex differentiable for every $n \in \mathbb{N}$. Using Theorem 3.7 it suffices to show that it has a differential at every point and that it satisifies the Cauchy–Riemann equations. Notice that since it is a polynomial (once expanded in terms of x and y and considered as map from $\mathbb{R}^2$ to $\mathbb{R}^2$ we trivially have that it has a differential). To see that it satisfies the Cauchy–Riemann equations, notice that (and similarly for v)

$$u_x = (Ref)_x = Re(f_x).$$

Therefore

$$f_x = u_x + \mathrm{i}v_x = n(x+\mathrm{i}y)^{n-1} \qquad\qquad f_y = u_y + \mathrm{i}v_y = n(x+\mathrm{i}y)^{n-1}\mathrm{i}.$$

Without computing what u_x, v_x, u_y, v_y are, notice that it follows from the expression above that

$$u_y + \mathrm{i}v_y = \mathrm{i}(u_x + \mathrm{i}v_x),$$

which implies that $u_x = v_y$ and $u_y = -v_x$, which are the Cauchy–Riemann equations.

3.2 Power Series

We want to focus on the study of power series, i.e. expressions of the form $\sum_{n=0}^{\infty} a_n z^n$. We begin by reviewing (in a very utilitarian way) some basic ideas of series for complex numbers covered in year 1.

Definition 3.9. *The series $\sum_{n=0}^{\infty} a_n$, with $a_n \in \mathbb{C}$ is convergent if and only if the sequence $S_N = \sum_{n=0}^{N} a_n$ is convergent in $\mathbb{C}$.*

Definition 3.10. *The series $\sum_{n=0}^{\infty}$, with $a_n \in \mathbb{C}$ is absolutely convergent if and only if the series $\sum_{n=0}^{\infty} |a_n|$ is convergent.*

The geometric series $\sum_{n=0}^{\infty} z^n$ is convergent if and only if $|z| < 1$, and sums up to $1/(1-z)$ (with partial sums $S_N = (1-z^{N+1})/(1-z)$). We review a couple of the convergence tests from year 1.

Theorem 3.11 (Ratio Test)**.** *Consider $\sum_{n=0}^{\infty} a_n$ and assume that $a_n \neq 0$ for all n. Then*

1. *If $\limsup \frac{|a_{n+1}|}{|a_n|} < 1$ then $\sum_{n=0}^{\infty} a_n$ is convergent.*
2. *If $\frac{|a_{n+1}|}{|a_n|} \geq 1$ for all $n > N$ then $\sum_{n=0}^{\infty} a_n$ is divergent.*

In particular if $\lim \frac{|a_{n+1}|}{|a_n|}$ exists, and equals L we have convergence for $L < 1$ and divergence for $L > 1$. (The test is inconclusive if $L = 1$.)

Theorem 3.12 (Root Test)**.** *Consider $\sum_{n=0}^{\infty} a_n$. Then*

1. *If $\limsup |a_n|^{1/n} < 1$ then $\sum_{n=0}^{\infty} a_n$ converges.*
2. *If $\limsup |a_n|^{1/n} > 1$ then $\sum_{n=0}^{\infty} a_n$ diverges.*

The proofs of these results are obtained by comparison with the geometric series and will not be covered in these notes.

We will focus on studying expressions of the form

$$\sum_{n=0}^{\infty} a_n z^n \qquad \text{or} \qquad \sum_{n=0}^{\infty} a_n (z-z_0)^n,$$

with $a_n, z \in \mathbb{C}$. The first observation is the existence of a radius of convergence.

Theorem 3.13. *Given $(a_n)_{n=0}^{\infty}$ there exists $R \in [0, \infty]$ such that*

$$\sum_{n=0}^{\infty} a_n z^n$$

converges for all $|z| < R$ and diverges for $|z| > R$. (As we will see in the proof $R = \frac{1}{\limsup |a_n|^{1/n}}$.)

Proof. We consider z given, but fixed, and apply the root test to the series given by $(a_n z^z)_{n=0}^{\infty}$. We know that the corresponding series is convergent if

$$\limsup |a_n z^n|^{1/n}$$

is less than 1 and divergent if it is greater than 1. But that translates in covergence if

$$|z| < \frac{1}{\limsup |a_n|^{1/n}}$$

and divergence when

$$|z| > \frac{1}{\limsup |a_n|^{1/n}},$$

proving the result. □

A simple application of the ratio tests yields the following result:

Theorem 3.14. *Let $a_n \neq 0$ for all $n \geq N$, and assume that $\lim \frac{|a_{n+1}|}{|a_n|}$ exists. Then $\sum_{n=0}^{\infty} a_n z^n$ has radius of convergence $R = \lim \frac{|a_n|}{|a_{n+1}|}$.*

Next we will show that within the radius of convergence a power series is actually differentiable, and that we can in fact compute the derivative term-by-term. More precisely:

Theorem 3.15. *Assume $\sum_{n=0}^{\infty} a_n z^n$ has radius of convergence R. Then for $|z| < R$ the function $f(z) = \sum_{n=0}^{\infty} a_n z^n$ is differentiable and*

$$f'(z) = \sum_{n=1}^{\infty} n a_n z^{n-1}.$$

Proof. First we will show that the power series for $f'(z)$ does have the same radius of convergence. Notice that the radius of convergence of $\sum_{n=1}^{\infty} n a_n z^{n-1}$ and $\sum_{n=1}^{\infty} n a_n z^n$ (i.e. where we have multiplied the expression by z) is the same. To see this notice that if $\sum_{n=1}^{\infty} n a_n z^{n-1}$ is convergent for $|z| < R$ and divergent for $|z| > R$ then the same will apply to the second series. Therefore we just need to consider

$$\limsup |n a_n|^{1/n} = \lim n^{1/n} \limsup |a_n|^{1/n} = \limsup |a_n|^{1/n},$$

which shows that the radius of convergence is the same as for $\sum_{n=0}^{\infty} a_n z^n$. Notice that the series $\sum_{n=2}^{\infty} n(n-1) a_n z^{n-2}$ also has the same radius of convergence (we will need this result in our estimate below, even though we never formally compute second derivatives).

Next, notice that for $k \in \mathbb{N}$ we have

$$\frac{w^k - z^k}{w - z} = w^{k-1} + w^{k-2} z + \cdots + w z^{k-2} + z^{k-1}. \tag{3.5}$$

Now, in order to prove that f is complex differentiable and compute its derivative we study

$$\frac{f(z+h) - f(z)}{h} - \sum_{n=1}^{\infty} n a_n z^{n-1}.$$

We denote by $w = z + h$ (and so $h = w - z$), and substitute the expression for f in terms of a series to find

$$\sum_{n=0}^{\infty} a_n \left(\frac{w^n - z^n}{w - z} - n z^{n-1} \right).$$

We look more carefully at the term in brackets. Using (3.5) we find (taking $k = n$)

$$\frac{w^n - z^n}{w - z} - n z^{n-1} = w^{n-1} + w^{n-2} z + \cdots + w z^{n-2} + z^{n-1} - n z^{n-1}$$

$$= w^{n-1} - z^{n-1} + [w^{n-2} - z^{n-2}]z + \cdots + (w-z)z^{n-2}$$

$$= (w-z)\left[\frac{w^{n-1} - z^{n-1}}{w-z} + \frac{w^{n-2} - z^{n-2}}{w-z}z + \cdots + \frac{w-z}{w-z}z^{n-2}\right]. \tag{3.6}$$

Now, for $|z| < r < R$ and $|w| < r < R$ we have

$$\left|\frac{w^k - z^k}{w-z}\right| = |w^{k-1} + w^{k-2}z + \cdots + wz^{k-2} + z^{k-1}| < kr^{k-1}$$

and therefore

$$\left|\frac{w^k - z^k}{w-z}z^{n-k-1}\right| \leq kr^{k-1}r^{n-k-1} \leq kr^{n-2},$$

which substituted in (3.6) yields

$$\leq |w-z|\left[|(n-1)r^{n-2} + (n-2)r^{n-2} + \cdots 2r^{n-2} + r^{n-2}|\right]$$

$$\leq |w-z|r^{n-2}\frac{1}{2}n(n-1).$$

We have shown that

$$\left|\frac{f(z+h) - f(z)}{h} - \sum_{n=1}^{\infty} na_n z^{n-1}\right| \leq |w-z|\frac{1}{2}\sum_{n=0}^{\infty} n(n-1)|a_n|r^{n-2} \leq M|h|,$$

which goes to zero as h goes to zero. Notice that in the last inequality we have used that the series $\sum_{n=0}^{\infty} n(n-1)|a_n|r^{n-2}$ is finite, since we observed that the radius of convergence of the corresponding power series was also R. □

We have the following simple consequence of the Theorem above, which allows us to compute the coefficients a_n in terms of derivatives of f.

Corollary 3.16. *Let $\sum_{n=0}^{\infty} a_n z^n$ be a power series with radius of convergence $R > 0$. Then $f(z) = \sum_{n=0}^{\infty} a_n z^n$ is infinitely differentiable and moreover*

$$f^{(n)}(0) = a_n n!, \qquad n = 0, 1, 2, \ldots$$

Proof. The result is trivial for $f(0)$, as it clearly equals a_0. A simple induction argument using the formula for the derivative of f in the previous Theorem yields the desired result. □

Theorem 3.17. *Let $\sum_{n=0}^{\infty} a_n z^n$ be a power series with radius of convergence $R > 0$. Then for every $r < R$ the sequence of functions*

$$f_k := \sum_{n=0}^{k} a_n z^n$$

converges uniformly in $|z| \leq r$.

Proof. We show the result by proving that (f_k) is uniformly Cauchy in $|z| \leq r$. We have (assuming that $j \leq k$)

$$|f_k(z) - f_j(z)| = |\sum_{n=j+1}^{k} a_n z^n| \leq \sum_{n=j+1}^{k} |a_n|r^n \leq \sum_{n=j+1}^{\infty} |a_n|r^n.$$

Since by assumption $\sum_{n=0}^{\infty} |a_n|r^n$ is finite, given any $\varepsilon > 0$ we can choose N large enough to make $|f_k(z) - f_j(z)| < \varepsilon$ for all $j, k > N$, concluding the proof. (This proof is essentially an application of the Weierstrass M-test that we covered a few weeks ago.) □

3.2.1 The exponential and the circular functions

Many of these functions should have appeared in year I, though perhaps only in the real-valued case.

Definition 3.18. *We define the following power series for $z \in \mathbb{C}$.*

$$\mathrm{e}^z := \sum_{n=0}^{\infty} \frac{1}{n!} z^n, \tag{3.7}$$

$$\cos(z) := \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} z^{2n}, \tag{3.8}$$

$$\cosh(z) := \sum_{n=0}^{\infty} \frac{1}{(2n)!} z^{2n}, \tag{3.9}$$

$$\sin(z) := \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} z^{2n+1}, \tag{3.10}$$

$$\sinh(z) := \sum_{n=0}^{\infty} \frac{1}{(2n+1)!} z^{2n+1}. \tag{3.11}$$

The ratio test shows (Exercise) that the radius of convergence of all of the series above is $R = \infty$. Notice that using Theorem 3.15 we can prove well known identities like $(\mathrm{e}^z)' = \mathrm{e}^z$. Indeed

$$(\mathrm{e}^z)' = \left(\sum_{n=0}^{\infty} \frac{1}{n!} z^n\right)' = \sum_{n=1}^{\infty} \frac{n}{n!} z^{n-1} = \sum_{n=0}^{\infty} \frac{1}{n!} z^n = \mathrm{e}^z.$$

In fact, we can easily relate all the circular functions to the exponential.

Proposition 3.19. *The following identities hold for all $z \in \mathbb{C}$:*

$$\cos z = \frac{\mathrm{e}^{\mathrm{i}z} + e^{-\mathrm{i}z}}{2}, \qquad \sin z = \frac{\mathrm{e}^{\mathrm{i}z} - e^{-\mathrm{i}z}}{2i},$$

$$\cosh z = \frac{\mathrm{e}^{z} + e^{-z}}{2}, \qquad \sinh z = \frac{\mathrm{e}^{z} - e^{-z}}{2}.$$

Proof. We only prove the first one. The others are very similar and are left as an Exercise.

$$\begin{aligned} \frac{\mathrm{e}^{\mathrm{i}z} + e^{-\mathrm{i}z}}{2} &= \frac{1}{2}\left[\sum_{n=0}^{\infty} \frac{1}{n!} (\mathrm{i}z)^n + \sum_{n=0}^{\infty} \frac{1}{n!} (-\mathrm{i}z)^n\right] \\ &= \frac{1}{2}\left[\sum_{n=0}^{\infty} \frac{\mathrm{i}^n + (-\mathrm{i})^n}{n!} z^n\right] = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} z^{2k}, \end{aligned}$$

where we have used that

$$\mathrm{i}^n + (-\mathrm{i})^n = \begin{cases} 2(\mathrm{i})^n = 2(-1)^{n/2} & n \text{ even} \\ 0 & n \text{ odd} \end{cases}.$$

□

There are additional relationships between sine and cosine and their hyperbolic counterparts. Notice that we have

$$\cos(\mathrm{i}z) = \cosh(z) \qquad \cosh(\mathrm{i}z) = \cos(z) \qquad \sin(\mathrm{i}z) = \mathrm{i}\sinh(z) \qquad \sinh(\mathrm{i}z) = \mathrm{i}\sin(z),$$

which shows that sine and cosine are unbounded functions in the complex plane. Just consider $z = \mathrm{i}y$ for $y \in \mathbb{R}$ together with the fact that the real valued $\sinh$ and $\cosh$ grow exponentially at infinity.

Theorem 3.20. *The exponential function* e^z *satisfies the following properties*

1. $\mathrm{e}^{z+w} = \mathrm{e}^z\mathrm{e}^w$ *for all* $z, w \in \mathbb{C}$.
2. $e^z \neq 0$ *for all* $z \in \mathbb{C}$.
3. $\mathrm{e}^z = 1$ *if and only if* $z = 2k\pi\mathrm{i}$ *for* $k \in \mathbb{Z}$, *and as a result* $\mathrm{e}^{z+w} = \mathrm{e}^z$ *if and only if* $w = 2k\pi\mathrm{i}$, $k \in \mathbb{Z}$. *Notice that in particular we have shown* $\mathrm{e}^{z+2k\pi\mathrm{i}} = \mathrm{e}^z$ *for all* $k \in \mathbb{Z}$, *so in this sense the exponential is periodic in the imaginary variable.*
4. $\mathrm{e}^z = -1$ *if and only if* $z = (2k+1)\pi\mathrm{i}$ *for* $k \in \mathbb{Z}$.

Proof. We present the direct proof of part 1, without using more advanced tools from complex analysis that would reduce the heavy computational nature.

$$\begin{aligned}\mathrm{e}^z\mathrm{e}^w = \left(\sum_{n=0} \frac{1}{n!}z^n\right)\left(\sum_{k=0}^{\infty} \frac{1}{k!}w^k\right) &= \sum_{l=0}^{\infty} \sum_{\substack{n,k\\ n+k=l}} \frac{z^n}{n!}\frac{w^k}{k!} \\ &= \sum_{l=0}^{\infty}\sum_{j=0}^{l} \frac{1}{l!}\binom{l}{j} z^j w^{l-j} = \sum_{l=0}^{\infty} \frac{1}{l!}(z+w)^l = \mathrm{e}^{z+w}.\end{aligned}$$

For part 2, notice that $\mathrm{e}^z e^{-z} = 1$, proving that $e^z \neq 0$. For part 3, denoting $z = x + \mathrm{i}y$ we find

$$\mathrm{e}^z = \mathrm{e}^x\mathrm{e}^{\mathrm{i}y} = \mathrm{e}^x(\cos y + \mathrm{i}\sin y),$$

which equals 1 if and only if $|\mathrm{e}^x| = 1$ and $\cos y + \mathrm{i}\sin y = 1$. These only happen if $x = 0$ and $y = 2\pi k$, $k \in \mathbb{Z}$. Similarly for part 4. □

3.2.2 Argument and Log

Every complex number $z \in \mathbb{C}\backslash\{0\}$ can be written in the form $z = |z|\mathrm{e}^{\mathrm{i}\theta}$, where θ is the angle that the vector z forms with the x axis, measured counter-clockwise. Of course that angle is not unique (but rather up to factors of 2π. Notice that for $z = 0$ there is no natural way to choose an angle.

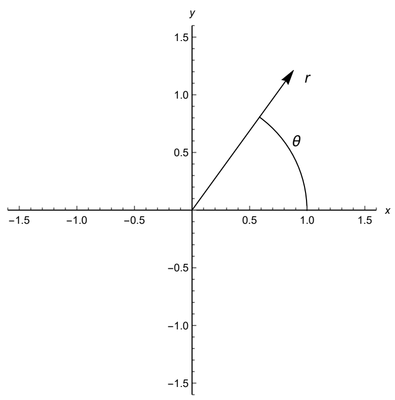

Figure 3.1: Polar representation of a complex number

We can define the (multivalued) function, for $z \neq 0$,

$$\arg(z) = \{\theta \in \mathbb{R} : z = |z|\mathrm{e}^{\mathrm{i}\theta}\}. \tag{3.12}$$

It is not a function as such, as the image is not uniquely defined, and if $\theta \in \arg(z)$ then so is $\theta + 2k\pi$. The following are easily verified properties of $\arg$.

Proposition 3.21.

1. $\arg(\alpha z) = \arg(z)$ *for all* $\alpha > 0$.
2. $\arg(\alpha z) = \arg(z) + \pi = \{\theta + \pi,$ *for* $\theta \in \arg(z)\}$ *for all* $\alpha < 0$,
3. $\arg(\bar{z}) = -\arg(z) = \{-\theta,$ *for* $\theta \in \arg(z)\}$,
4. $\arg(1/z) = -\arg(z)$,
5. $\arg(zw) = \arg(z) + \arg(w) = \{\theta + \phi,$ *with* $\theta \in \arg(z), \phi \in \arg(w)\}$.

The ambiguity of the argument function can be solved by defining the principal value Arg of the $\arg$ function to take values in $(-\pi, \pi]$. That is for any $z \in \mathbb{C}$ we have $\mathrm{Arg}(z) \in (-\pi, \pi]$.

Notice that it is impossible to define the Arg function continuously in the entire plane. In particular as we approach any point in the negative real axis, if we do it from above the Arg function will yield π, while if we do it from below it will $-\pi$. Observe that if we had made any other choice for the range of Arg there would always be a half-line where we have the same issue, the difference between the values of the argument when approaching from opposite sides is always 2π.

We want to define the logarithm by analogy of what happens in $\mathbb{R}$. In the real valued case we say (here $w, z \in \mathbb{R}$)

$$w = \log(z) \quad \text{if and only if} \quad \mathrm{e}^{w} = z.$$

If we could extend this for $w, z \in \mathbb{C}$, since we know that $\mathrm{e}^{w} = \mathrm{e}^{w+2\pi \mathrm{i} k}$ for any $k \in \mathbb{Z}$ we would have that if $w = \log(z)$ the so is $w + 2\pi \mathrm{i} k$. Therefore we would have that $\log(z)$ is a multivalued function (just like it happened before with $\arg(z)$, the argument function).

Let's write $z = |z|\mathrm{e}^{\mathrm{i}\arg(z)}$ and $w = \log(z) = u + \mathrm{i}v$. We have

$$\mathrm{e}^{u+\mathrm{i}v} = \mathrm{e}^{u}\mathrm{e}^{\mathrm{i}v} = z = |z|\mathrm{e}^{\mathrm{i}\arg(z)},$$

and therefore comparing the two expressions in polar form we must have

$$\mathrm{e}^{u} = |z| \qquad \text{and} \qquad \mathrm{e}^{\mathrm{i}v} = \mathrm{e}^{\mathrm{i}\arg(z)}.$$

That means that $u = \log|z|$, with this logarithm being the *real* logarithm. We will denote by Log the logarithm in $\mathbb{R}$ to distinguish it from the complex valued we want to define. We define the multivalued function

$$\log(z) = \mathrm{Log}|z| + \mathrm{i}\arg(z). \tag{3.13}$$

In terms of the Arg function we have

$$\log(z) = \mathrm{Log}|z| + \mathrm{i}\mathrm{Arg}(z) + 2\pi\mathrm{i}k \qquad \text{for } k \in \mathbb{Z}.$$

For example if we compute the complex logarithm of 1 we have

$$\log(1) = \mathrm{Log}|1| + \mathrm{i}\mathrm{Arg}(1) + 2\pi\mathrm{i}k = 2\pi\mathrm{i}k \qquad \text{for } k \in \mathbb{Z}.$$

Notice that the definition above makes sense provided that $z \neq 0$, where the *real* logarithm is not defined. We can now compute logarithms of negative numbers.

$$\log(-1) = \mathrm{Log}|-1| + \mathrm{i}\mathrm{Arg}(-1) + 2\pi\mathrm{i}k = \mathrm{i}\pi + 2\pi\mathrm{i}k \qquad \text{for } k \in \mathbb{Z}.$$

The complex logarithm we have just defined obeys many of the properties that we know for the real logarithm, with the caveat that we have to take care of the multi-valuedness of the function. For example

$$\log(zw) = \log z + \log w.$$

To prove this result, notice that since $\mathrm{Log}|zw| = \mathrm{Log}(|z||w|) = \mathrm{Log}|z| + \mathrm{Log}|w|$ and that $\arg(zw) = \arg(z) + \arg(w)$ we have

$$\log(zw) = \mathrm{Log}|zw| + \mathrm{i}\arg(zw) = \mathrm{Log}|z| + \mathrm{Log}|w| + \mathrm{i}\arg(z) + \mathrm{i}\arg(w) = \log z + \log w.$$

This equality needs to be understood *modulo* $2\pi\mathrm{i}$, that is there exists $k \in \mathbb{Z}$ such that

$$\log(zw) - \log(z) - \log(w) = 2\pi\mathrm{i}k.$$

Similarly we have

$$\log(z/w) = \log(z) - \log(w).$$

If we want to consider the (complex) differentiability of the $\log$ we have to deal with the multi-valuedness of the $\arg$ function. Indeed if we consider the incremental quotient

$$\frac{\log(z + \Delta z) - \log z}{\Delta z}$$

we need to make sure that as we approach z both $\log$s approach the same value. We know that this cannot be done continuously in the entire plane, and that we need to remove a semi-line arising from the origin. For example if we consider $\mathbb{C}\backslash\{x \leq 0\}$ we can consider the **principal branch** of the logarithm, which by an abuse of notation we denote by Log, just like the real logarithm, by

$$\mathrm{Log}(z) = \mathrm{Log}|z| + \mathrm{i}\mathrm{Arg}(z).$$

This function, defined on $\mathbb{C}\backslash\{x \leq 0\}$ is single valued. If we consider points of the form $z = x \pm \mathrm{i}\varepsilon$, for $x < 0$ and $\varepsilon > 0$ small, we find

$$\lim_{\varepsilon \to 0} \mathrm{Log}(x \pm \mathrm{i}\varepsilon) = \mathrm{Log}(x) \pm \mathrm{i}\pi,$$

showing that the function could not be extended continuously along $\{x < 0\}$. This half-line is called a **branch cut**. It is possible to compute the derivative of Log directly from the definition, or in terms of its inverse. However, for practical purposes, once we know it is differentiable, from the identity

$$\mathrm{e}^{\mathrm{Log}z} = z$$

we find

$$e^{\mathrm{Log}z}(\mathrm{Log}z)' = 1$$

from which it follows that $(\mathrm{Log}z)' = 1/z$.

Once we have defined the notion of logarithm it is possible to consider defining complex powers of complex numbers. Given $\alpha \in \mathbb{C}$, and $z \neq 0$ we define the α-th power of z by

$$z^\alpha := \mathrm{e}^{\log(z^\alpha)} = \mathrm{e}^{\alpha\mathrm{Log}|z| + \alpha\mathrm{i}\arg(z)}.$$

The multi-valuedness of arg means that the same is true for z^α. If we rewrite the above as

$$z^\alpha = \mathrm{e}^{\alpha\mathrm{Log}|z| + \alpha\mathrm{i}\arg(z)} = \mathrm{e}^{\alpha\mathrm{Log}|z| + \alpha\mathrm{i}\mathrm{Arg}(z) + 2\pi\alpha k\mathrm{i}} = \mathrm{e}^{\alpha\mathrm{Log}(z)}\mathrm{e}^{2\pi\alpha k\mathrm{i}}$$

for $k \in \mathbb{Z}$ the multi-valuedness becomes more evident. The number of α powers, whether it is one, finitely many or infinitely many will depend on α.

Indeed if α is an integer for example then $\mathrm{e}^{2\pi\alpha k\mathrm{i}} = 1$, which means that in fact there is only one value of z^α. If α is rational, say $\alpha = p/q$, with p, q coprime, then z^α will have finitely many powers. It is easy to see that for $\alpha = p/q$ (with p, q coprime, and $q \in \mathbb{N}$)

$$\mathrm{e}^{2\pi\alpha k\mathrm{i}} = \mathrm{e}^{2\pi\alpha(k+q)\mathrm{i}}$$

and therefore z^α will take q different values

$$\mathrm{e}^{\alpha \mathrm{Log}(z)}\mathrm{e}^{2\pi\alpha k\mathrm{i}}, \qquad k = 0, 1, \dots, q-1.$$

In the case of an irrational α it will actually take infinitely many values.

In the rational case the result obtained above is consistent with what we know about finding roots of polynomials. If we consider, for $q \in \mathbb{N}$ the equation $z^q = 1$ we know it should have q roots which correspond to

$$z = 1^{1/q}.$$

Now, using the expressions above we find

$$1^{1/q} = \mathrm{e}^{\mathrm{Log}(1)/q}\mathrm{e}^{2\pi\mathrm{i}k/q} = \mathrm{e}^{2\pi\mathrm{i}k/q} \qquad k = 0, 1, \dots, q-1.$$

3.3 Complex integration, contour integrals

For a function $f : [a, b] \to \mathbb{C}$ we define

$$\int_a^b f(t)\mathrm{d}t = \int_a^b \mathbf{Re}\, f(t)\mathrm{d}t + \mathrm{i}\int_a^b \mathbf{Im}\, f(t)\mathrm{d}t. \tag{3.14}$$

This definition means that we reduce integrating a complex-valued function to integrating two real-valued functions, and can therefore use every result we know from before, such as the Fundamental Theorem of Calculus to compute each integral.

It is easy to see that for every $f, g : [a, b] \to \mathbb{C}$ and every $\alpha, \beta \in \mathbb{C}$ we have

$$\int_a^g [\alpha f + \beta g]\mathrm{d}t = \alpha\int_a^b f(t)\mathrm{d}t + \beta\int_a^b g(t)\mathrm{d}t.$$

That $\int_a^b (f+g)\mathrm{d}t = \int_a^b f\mathrm{d}t + \int_a^b g\mathrm{d}t$ follows immediately from the definition. We show the more tedious $\int_a^b \alpha f(t)\mathrm{d}t = \alpha\int_a^b f(t)\mathrm{d}t$. We have (suppressing the limits of integration and $\mathrm{d}t$ for simplicity)

$$\alpha\int f = \alpha\left[\int \mathbf{Re}(f) + \mathrm{i}\int \mathbf{Im}(f)\right]$$

$$= \mathbf{Re}(\alpha)\int \mathbf{Re}(f) - \mathbf{Im}(\alpha)\int \mathbf{Im}(f) + \mathrm{i}\left[\mathbf{Im}(\alpha)\int \mathbf{Re}(f) + \mathbf{Re}(\alpha)\int \mathbf{Im}(f)\right]$$

$$= \int \mathbf{Re}(\alpha)\,\mathbf{Re}(f) - \mathbf{Im}(\alpha)\,\mathbf{Im}(f) + \mathrm{i}\left[\int \mathbf{Im}(\alpha)\,\mathbf{Re}(f) + \mathbf{Re}(\alpha)\,\mathbf{Im}(f)\right]$$

$$= \int \mathbf{Re}(\alpha f) + \mathrm{i}\int \mathbf{Im}(\alpha f) = \int (\alpha f).$$

Notice that in this case

$$\overline{\int_a^b f(t)\mathrm{d}t} = \int_a^b \overline{f(t)}\mathrm{d}t. \tag{3.15}$$

Indeed

$$\overline{\int_a^b f(t)\mathrm{d}t} = \int_a^b \mathbf{Re}\, f(t)\mathrm{d}t - \mathrm{i}\int_a^b \mathbf{Im}\, f(t)\mathrm{d}t = \int_a^b \mathbf{Re}\,\overline{f(t)}\mathrm{d}t + \mathrm{i}\int_a^b \mathbf{Im}\,\overline{f(t)}\mathrm{d}t = \int_a^b \overline{f(t)}\mathrm{d}t.$$

We also have the following estimate (which we will use repeteadly below)

$$\left|\int_a^b f(t)\mathrm{d}t\right| \le \int_a^b |f(t)|\mathrm{d}t. \tag{3.16}$$

To prove this result, assume that $\int_a^b f(t)\mathrm{d}t = R\mathrm{e}^{\mathrm{i}\theta}$, where $R = \left|\int_a^b f(t)\mathrm{d}t\right|$. As a result of this representation R also equals

$$R = \mathrm{e}^{-\mathrm{i}\theta}\int_a^b f(t)\mathrm{d}t = \int_a^b \mathrm{e}^{-\mathrm{i}\theta} f(t)\mathrm{d}t.$$

Now, if we write $\mathrm{e}^{-\mathrm{i}\theta} f(t) = u + \mathrm{i}v$, with u and v real valued. Then we must have

$$R = \int_a^b u\mathrm{d}t \qquad \int_a^b v\mathrm{d}t = 0.$$

Notice that $u = \mathbf{Re}[\mathrm{e}^{-\mathrm{i}\theta} f(t)] \leq \left|\mathrm{e}^{-\mathrm{i}\theta} f(t)\right| \leq |f(t)|$. This implies

$$R = \int_a^b u(t)\mathrm{d}t \leq \int_a^b |f(t)|\mathrm{d}t.$$

But since R equals $\left|\int_a^b f(t)\mathrm{d}t\right|$ we are done.

The definition above is a natural choice for integrating functions from $\mathbb{R}$ to $\mathbb{C}$, with a far less obvious choice for integrating a function from $\mathbb{C}$ to $\mathbb{C}$. Instead, we want to study integrals of complex valued-valued functions along curves, that is, expressions of the form

$$\int_\Gamma f\mathrm{d}z,$$

where Γ is curve in the complex plane. To define a curve in $\mathbb{C}$, consider a function $\gamma : [a, b] \to \mathbb{C}$, given by $\gamma(t) = x(t) + \mathrm{i}y(t)$. We will ask that the curve γ be C^1. The primary reason is that we want to have a well defined tangent at every point of the curve (which is also integrable). We say that the curve $\Gamma = \gamma([a, b]) \subset \mathbb{C}$ is *parametrised* by the map γ.

Definition 3.22. *Given a function $f : \Omega \subset \mathbb{C} \to \mathbb{C}$ along the path $\Gamma \subset \Omega \subset \mathbb{C}$ parametrised by $\gamma : [a, b] \to \mathbb{C}$ the integral of f over Γ is given by*

$$\int_\Gamma f\mathrm{d}z = \int_a^b f(\gamma(t))\gamma'(t)\mathrm{d}t = \int_a^b \mathbf{Re}(f(\gamma(t))\gamma'(t))\mathrm{d}t + \mathrm{i}\int_a^b \mathbf{Im}(f(\gamma(t))\gamma'(t))\mathrm{d}t.$$

Notice that we are not making any regularity assumptions on f, just that the integrals are well defined. Sometimes we will consider more than one parametrisation of a curve Γ, say γ_1 and γ_2 and will use the notation $\int_{\gamma_1} f$ and $\int_{\gamma_2} f$ in addition to $\int_\Gamma$.

On many occasions we want to consider curves that are not C^1 but perhaps just piece-wise C^1. For example a square. In this case we can think of Γ as a union of n curves Γ_j, each one C^1, and parametrised in the right direction, so that connected in the right order they describe the entire curve Γ. We can define

$$\int_\Gamma f\,\mathrm{d}z := \sum_{j=1}^n \int_{\Gamma_j} f\,\mathrm{d}z.$$

It is straight forward from the definition (details are left as an Exercise) that given a curve Γ, and two functions $f, g : \mathbb{C} \to \mathbb{C}$ and $\alpha, \beta \in \mathbb{C}$ we have

$$\int_\Gamma (\alpha f(z) + \beta g(z))\mathrm{d}z = \alpha\int_\Gamma f(z)\mathrm{d}z + \beta\int_\Gamma g(z)\mathrm{d}z.$$

If we allow for $\gamma'(t)$ not to exists at finitely many points, this can be defined as a single integral, with clearly both formulations being equivalent.

Example 3.23. *Let $f : \mathbb{C} \to \mathbb{C}$ be given by $f(z) = f(x + \mathrm{i}y) = x^4 + \mathrm{i}y^4$ and the curve joining the origin in a straight line to the point $1 + \mathrm{i}$, parametrized by $\gamma : [0, 1] \to \mathbb{C}$, $\gamma(t) = (1 + \mathrm{i})t$. Notice that $\gamma'(t) = 1 + \mathrm{i}$ and so we have*

$$\int_\Gamma f = \int_0^1 (t^4 + \mathrm{i}t^4)(1 + \mathrm{i})\mathrm{d}t = \int_0^1 2\mathrm{i}t^4\mathrm{d}t = \frac{2}{5}\mathrm{i}.$$

In the next Lemma we want to show that $\int_\Gamma f$ depends only on the orientation of the parametrisation of the curve. More precisely

Lemma 3.24. *Let Γ be a curve in $\mathbb{C}$, parametrised by $\gamma : [a,b] \to \mathbb{C}$, that is $\gamma([a,b]) = \Gamma$. Given $f : \Omega \subset \mathbb{C} \to \mathbb{C}$ and $\Gamma \subset \Omega$ we have:*

1. *if γ^- represents the parametrisation of γ in the opposite direction, then*

$$\int_{\gamma^-} f = -\int_\gamma f.$$

If a curve Γ has attached a sense of direction we will call it a directed curve. In this case we will denote by $-\Gamma$ the same curve swept in the opposite direction. Without the need to specify the parametrisation we can reformulate the above result by

$$\int_\Gamma f \mathrm{d}z = -\int_{-\Gamma} f \mathrm{d}z.$$

2. *If $\tilde{\gamma} : [\tilde{a}, \tilde{b}] \to \mathbb{C}$ is another parametrisation of Γ that preserves the orientation then*

$$\int_{\tilde{\gamma}} f = \int_\gamma f.$$

We refer to this fact as reparametrisation invariance. [In practise, with the regularity we are demanding on the curves, this means that there exists $\phi : [\tilde{a}, \tilde{b}] \to [a,b]$, bijective and increasing, such that $\tilde{\gamma} = \gamma(\phi)$.]

Proof. 1. Notice that if $\gamma : [a,b] \to \mathbb{C}$ parametrises the curve in one direction then γ^- is given by $\gamma^- : [a,b] \to \mathbb{C}$ with $\gamma^-(t) = \gamma(a+b-t)$. Therefore

$$\int_{\gamma^-} f = \int_a^b f(\gamma^-(t)) \left(\gamma^-\right)'(t)\mathrm{d}t = \int_a^b f(\gamma(a+b-t))(-\gamma'(a+b-t))\mathrm{d}t$$

$$= \int_b^a f(\gamma(s))(-\gamma'(s))(-1)\mathrm{d}s = -\int_a^b f(\gamma(s))\gamma'(s)\mathrm{d}s = -\int_\gamma f.$$

2. The proof is very similar to part one.

$$\int_{\tilde{\gamma}} f = \int_{\tilde{a}}^{\tilde{b}} f(\tilde{\gamma}(t))\tilde{\gamma}'(t)\mathrm{d}t = \int_{\tilde{a}}^{\tilde{b}} f(\gamma(\phi(t)))\gamma'(\phi(t))\phi'(t)\mathrm{d}t = \int_a^b f(\gamma(s))\gamma'(s)\mathrm{d}s = \int_\gamma f,$$

where we have made the change of variables $\phi(t) = s$ and therefore $\phi'(t)\mathrm{d}t = \mathrm{d}s$ ☐

Consider the function $f(z) = 1$ as a complex-valued function and a curve $\gamma : [a,b] \to \mathbb{C}$. Then

$$\int_\gamma f \mathrm{d}z = \int_a^b \gamma'(t)\mathrm{d}t.$$

Here $\gamma'(t)$ is a complex valued number and the integral will be a complex number. For example if we take γ just like in Example 3.23 we have $\gamma'(t) = 1 + \mathrm{i}$ and $\int_\gamma f \mathrm{d}z = \int_0^1 (1+\mathrm{i})\mathrm{d}t = 1 + \mathrm{i}$. This is because we are considering $\mathrm{d}z$ as complex valued, given by $\gamma'(t)\mathrm{d}t$.

We could consider defining the integral

$$\int_\gamma |\mathrm{d}z| := \int_a^b |\gamma'(t)|\mathrm{d}t = \int_a^b \sqrt{(x'(t))^2 + (y'(t))^2}\mathrm{d}t = l(\gamma),$$

where $\gamma : [a,b] \to \mathbb{C}$ is given by $\gamma(t) = x(t) + \mathrm{i}y(t)$, and $l(\gamma)$ stands for the length of the curve γ.

Similarly for $f : \mathbb{C} \to \mathbb{C}$ we can define

$$\int_\gamma |f||\mathrm{d}z| := \int_a^b |f(\gamma(t)||\gamma'(t)|\mathrm{d}t.$$

Notice that $\int_\gamma |f||\mathrm{d}z| \geq 0$ and that we have

$$\left|\int_\gamma f\mathrm{d}z\right| \leq \int_\gamma |f||\mathrm{d}z|.$$

To show this notice that (using (3.16))

$$\left|\int_\gamma f\mathrm{d}z\right| = \left|\int_a^b f(\gamma(t))\gamma'(t)\mathrm{d}t\right| \leq \int_a^b |f(\gamma(t)||\gamma'(t)|\mathrm{d}t = \int_\gamma |f||\mathrm{d}z|.$$

We can further estimate the right-hand side

$$\int_\gamma |f||\mathrm{d}z| \leq \max_{z\in\Gamma}|f(z)| \int_\gamma |\mathrm{d}z| = \max_{z\in\Gamma}|f(z)|l(\gamma).$$

Therefore we obtain

$$\left|\int_\gamma f\mathrm{d}z\right| \leq \max_{z\in\Gamma}|f(z)|l(\gamma).$$

Definition 3.25. *Given $f : \mathbb{C} \to \mathbb{C}$ and a curve $\gamma : [a, b] \to \mathbb{C}$ we define*

$$\int_\gamma f\mathrm{d}\bar{z} := \int_a^b f(\gamma(t))\overline{\gamma'(t)}\mathrm{d}t.$$

Observe that in general

$$\overline{\int_\gamma f(z)\mathrm{d}z}$$

is not equal to

$$\int_\gamma \overline{f(z)}\mathrm{d}z,$$

unlike when we considered functions $f : [a, b] \to \mathbb{C}$; see (3.15). Instead we have

$$\overline{\int_\gamma f(z)\mathrm{d}z} = \overline{\int_a^b f(\gamma(t))\gamma'(t)\mathrm{d}t} = \int_a^b \overline{f(\gamma(t))\gamma'(t)}\mathrm{d}t = \int_a^b \overline{f(\gamma(t))}\,\overline{\gamma'(t)}\mathrm{d}t = \int_\gamma \overline{f(z))}\mathrm{d}\bar{z}.$$

We compute a few more examples of integrals along curves.

Example 3.26. *Integrate $f(z) = \bar{z}$ (the definition does not require functions to be analytic) along the circle of centred at $1 + \mathrm{i}$ of radius 2 (oriented counterclockwise).*

First we describe the curve γ. Notice that $2\mathrm{e}^{\mathrm{i}t}$ for $t \in [0, 2\pi)$ describes a circle or raidus two centred at the origin and with the required orientation. Therefore $\gamma(t) = (1 + \mathrm{i}) + 2\mathrm{e}^{\mathrm{i}t}$ for $t \in [0, 2\pi)$. We have $\gamma'(t) = 2\mathrm{i}\mathrm{e}^{\mathrm{i}t}$. Therefore the integral becomes

$$\int_\gamma f(z)\mathrm{d}z = \int_0^{2\pi} \overline{((1 + \mathrm{i}) + 2\mathrm{e}^{\mathrm{i}t})}2\mathrm{i}\mathrm{e}^{\mathrm{i}t}\mathrm{d}t = 2(1 - \mathrm{i})\mathrm{i}\int_0^{2\pi} \mathrm{e}^{\mathrm{i}t}\mathrm{d}t + \int_0^{2\pi} 4i = 8\pi\mathrm{i},$$

since $\int_0^{2\pi} \mathrm{e}^{\mathrm{i}t}\mathrm{d}t = 0$. Indeed

$$\int_0^{2\pi} \mathrm{e}^{\mathrm{i}t}\mathrm{d}t = \int_0^{2\pi} \cos t\mathrm{d}t + \mathrm{i}\int_0^{2\pi} \sin t\mathrm{d}t = 0.$$

In fact

$$\int_0^{2\pi} \mathrm{e}^{\mathrm{i}nt}\mathrm{d}t = 0, \qquad \text{for all } n \neq 0.$$

Example 3.27. *Integrate $f(z) = z$ along the circle of centred at $1+\mathrm{i}$ of radius 2 (oriented counterclockwise). As before $\gamma(t) = (1+\mathrm{i}) + 2\mathrm{e}^{\mathrm{i}t}$ for $t \in [0, 2\pi)$. We have $\gamma'(t) = 2\mathrm{i}\mathrm{e}^{\mathrm{i}t}$. Therefore the integral becomes*

$$\int_\gamma f(z)\mathrm{d}z = \int_0^{2\pi} \Big((1+\mathrm{i}) + 2\mathrm{e}^{\mathrm{i}t}\Big)2\mathrm{i}\mathrm{e}^{\mathrm{i}t}\mathrm{d}t = 2(1+\mathrm{i})\mathrm{i}\int_0^{2\pi} \mathrm{e}^{\mathrm{i}t}\mathrm{d}t + \int_0^{2\pi} 4i\mathrm{e}^{\mathrm{i}t}\mathrm{d}t = 0,$$

using that $\int_0^{2\pi} \mathrm{e}^{\mathrm{i}nt}\mathrm{d}t = 0,$ for all $n \neq 0$.

Theorem 3.28. *Assume that $F : \Omega \subset \mathbb{C} \to \mathbb{C}$ is analytic (Ω open) and set $f(z) = \frac{\mathrm{d}F}{\mathrm{d}z}$, with f continuous. Let $\gamma : [a, b] \to \Omega$ be a C^1 curve. Then*

$$\int_\gamma f\mathrm{d}z = F(\gamma(b)) - F(\gamma(a)).$$

Proof. We have

$$\int_\gamma f\mathrm{d}z = \int_a^b f(\gamma(t))\gamma'(t)\mathrm{d}t = \int_a^b \frac{\mathrm{d}F}{\mathrm{d}z}(\gamma(t))\gamma'(t)\mathrm{d}t = \int_a^b \frac{\mathrm{d}}{\mathrm{d}t}F(\gamma(t))\mathrm{d}t = F(\gamma(b)) - F(\gamma(a)).$$

□

We remark that there are no assumptions made about Ω other than it is open. That is, all we need for the result to be true, is that f is analytic in an open neighborhood of the curve. The notion of simply connected (for a domain) will be defined later, but we emphasize that there is no such requirement on Ω above result to be true.

3.3.1 Links with MA259

We want to connect the notion of contour integral with the notions introduced in MA259. We begin by recalling the objects introduced there (with their notation).

The line integral we have just defined has many similarities the with notion of tangential line integral introduced in MA259 for a vectorfield $\underline{v}$. There the definition read

$$\int_{C_{pq}} \underline{v} \cdot dr := \int_\alpha^\beta \underline{v}(r(t)) \cdot \frac{dr}{dt}dt,$$

where C_{pq} is a curve parametrised by $r : [\alpha, \beta] \to \mathbb{R}^n$ with $r(\alpha) = p$ and $r(\beta) = q$. For closed curves that integral is usually referred as circulation.

Another integral arising in MA259 is the flux integral, which is given by

$$\int_C \underline{v} \cdot N\mathrm{d}t.$$

Here N represents the normal, with the following convention. If the curve C is parametrised by $r(t) = (x(t), y(t))$, and $r'(t) = (x'(t), y'(t))$ has the same direction of the tangent, we choose

$$\underline{N}(t) := r'(t)^\perp = (y'(t), -x'(t)).$$

When considering the curves determining the boundary of a regular domain we will consider them as positively oriented. That is, choose the orientation so that the corresponding $\underline{N}$ as defined above corresponds (i.e. has the same direction as) to the outward normal.

The following results (considered here only for two dimensions) correspond to Green's and Gauss' Theorems. For a positively oriented regular domain Ω we have

$$\iint_\Omega \mathrm{curl}\underline{v}\mathrm{d}x\mathrm{d}y = \oint_{\partial\Omega} \underline{v} \cdot \mathrm{d}r$$

and

$$\iint_\Omega \mathrm{div}\underline{v}\mathrm{d}x\mathrm{d}y = \oint_{\partial\Omega} \underline{v} \cdot N\mathrm{d}t.$$

Now let's consider out contour integral $\int_\gamma f(z)\mathrm{d}z$ for a function $f = u + \mathrm{i}v$ and a curve $\gamma(t) = \gamma_1(t) + \mathrm{i}\gamma_2(t)$. We have

$$\int_\gamma f(z)\mathrm{d}z = \int_a^b [u(\gamma(t)) + \mathrm{i}v(\gamma(t))][\gamma_1'(t) + \mathrm{i}\gamma_2'(t)]\mathrm{d}t$$

$$= \int_a^b u(\gamma(t))\gamma_1'(t) - v(\gamma(t))\gamma_2'(t)\mathrm{d}t + \mathrm{i}\int_a^b u(\gamma(t))\gamma_2'(t) + v(\gamma(t))\gamma_1'(t)\mathrm{d}t$$

$$= \int_a^b (u, -v) \cdot (\gamma_1', \gamma_2')\mathrm{d}t + \mathrm{i}\int_a^b (u, -v) \cdot (\gamma_2', -\gamma_1')\mathrm{d}t$$

$$= \int_\gamma (u, -v) \cdot \mathrm{d}r + \mathrm{i}\int_\gamma (u, -v) \cdot N\mathrm{d}t,$$

and so if we define the vector field $\underline{f} = (u, -v)$, we have just shown that

$$\int_\gamma f\mathrm{d}z = \text{circulation}(\underline{f}) + \mathrm{i}\,\text{flux}(\underline{f}).$$

Using the above expression, together with Green's and Gauss' Theorem we can prove the following result.

Theorem 3.29 (Cauchy's Theorem)**.** *Let $f : \Omega \to \mathbb{C}$ be an analytic function, with Ω an open, simply connected domain. Let γ be a C^1 closed curve in Ω. Then*

$$\int_\gamma f(z)\mathrm{d}z = 0.$$

Before we prove the result we define simply connected. Loosely speaking means that the domain contains no holes. A set of more formal definitions is as follows.

Definition 3.30. *A set $\Omega \subset \mathbb{C}$ is connected if it cannot be expressed as the union of non-empty open sets Ω_1 and Ω_2 such that $\Omega_1 \cap \Omega_2 = \emptyset$. An open, connected set $\Omega \subset \mathbb{C}$ is called simply connected if every closed curve in Ω can be continuously deformed to a point.*

Proof. The proof presented here assumes that the curve a simple, regular curve and that f' is continuous. If the domain is simply connected, the region inside the curve does not have any holes, and f is analytic in it. We know

$$\int_\gamma f\mathrm{d}z = \text{circulation}(\underline{f}) + \mathrm{i}\,\text{flux}(\underline{f})$$

$$= \iint_\Omega \mathrm{curl}\underline{v}\mathrm{d}x\mathrm{d}y + \mathrm{i}\iint_\Omega \mathrm{div}\underline{v}\mathrm{d}x\mathrm{d}y.$$

We claim that both terms are actually 0, because $\mathrm{curl}\underline{f} = \mathrm{div}\underline{f} = 0$. Since $\underline{f} = (u, -v)$ we have

$$\mathrm{div}\underline{f} = u_x - v_y \qquad \mathrm{curl}\underline{f} = -v_x - u_y$$

but since $f = u + \mathrm{i}v$ is analytic it satisfies the Cauchy–Riemann equations,

$$u_x = v_y \qquad v_x = -u_y$$

which imply the result. ☐

Notice that Cauchy's Theorem applies to Example 3.27, where the function is analytic, but obviously not to Example 3.26, where the function is not analytic.

Cauchy's Theorem works for more general *curves*. Consider the shaded region Ω in Figure 3.2. If we think of its boundary as a *one curve* Γ, even though it is formed by two separate curves we have

$$\int_\Gamma f \mathrm{d}z = 0,$$

provided that Γ is oriented positively. That means that the exterior curve, that we denote by γ_1 needs to be oriented counter-clockwise, while the interior curve, denoted by γ_2 has to be oriented clockwise.

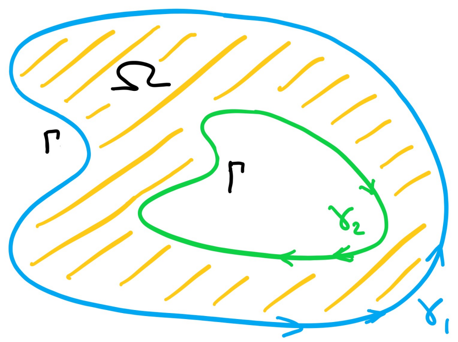

Figure 3.2: Region bound by two positively oriented curves

An equivalent formulation of this fact, which will be extremely useful is known as the deformation of contour Theorem.

Theorem 3.31. *Let $\Omega \subset \mathbb{C}$ be a region bounded by two simple curves γ_1 (the exterior curve) and γ_2 (the interior). Assume they are oriented positively, and let f be an analytic function in $\Omega \cup \gamma_1 \cup \gamma_2$. Then*

$$\int_{\gamma_1} f \mathrm{d}z + \int_{\gamma_2} f \mathrm{d}z = 0.$$

If we denote by γ_2^- the anti-clockwise parametrization of γ_2, then the result can be rephrased as

$$\int_{\gamma_1} f \mathrm{d}z = \int_{\gamma_2^-} f \mathrm{d}z,$$

that is the integral is the same along both curves when both are parametrised counter-clockwise.

Proof. The proof is based on creating two new contours of integration, the boundaries of two simply connected regions where f is analytic so that we can apply Cauchy's Theorem 3.29.

To achieve this we add two new curves to the previous picture, now in yellow in Figure 3.3. They join the points A (in $\gamma 1$) with D (in γ_2) and the points B (in $\gamma 1$) with C (in γ_2). The two curves we want to consider are denoted by ρ and η. Each one of them is piecewise C^1 and formed by four sections. Each one of these curves is oriented positively with respect to the region they enclose, that is, they are both oriented counter-clockwise.

By Cauchy's Theorem

$$\int_\rho f \mathrm{d}z = \int_{\rho_1} f \mathrm{d}z + \int_{\rho_2} f \mathrm{d}z + \int_{\rho_3} f \mathrm{d}z + \int_{\rho_4} f \mathrm{d}z = 0, \tag{3.17}$$

$$\int_\eta f \mathrm{d}z = \int_{\eta_1} f \mathrm{d}z + \int_{\eta_2} f \mathrm{d}z + \int_{\eta_3} f \mathrm{d}z + \int_{\eta_4} f \mathrm{d}z = 0. \tag{3.18}$$

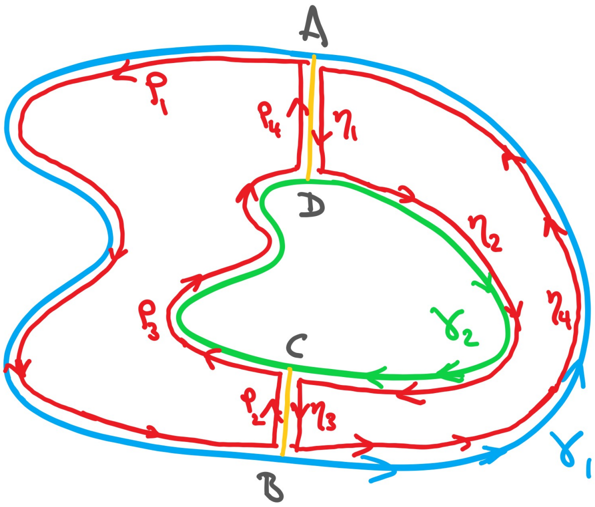

Figure 3.3: two positively oriented curves

We observe that η_1 and ρ_4 correspond to the same curve but with parametrisations in opposite directions. Similarly for η_3 and ρ_2. Therefore

$$\int_{\eta_1} f\mathrm{d}z + \int_{\rho_4} f\mathrm{d}z = 0 \qquad\qquad \int_{\eta_3} f\mathrm{d}z + \int_{\rho_2} f\mathrm{d}z = 0.$$

Adding (3.17) and (3.18) and using the above identities we find

$$\int_{\rho_1} f\mathrm{d}z + \int_{\rho_3} f\mathrm{d}z + \int_{\eta_2} f\mathrm{d}z + \int_{\eta_4} f\mathrm{d}z = 0$$

Also notice that ρ_1 and η_4 together build γ_1, while ρ_3 and η_2 build γ_2. Therefore, the above equality can be rewritten as

$$\int_{\gamma_1} f\mathrm{d}z + \int_{\gamma_2} f\mathrm{d}z = 0.$$

Since

$$\int_{\gamma_2} f\mathrm{d}z = -\int_{\gamma_2^-} f\mathrm{d}z$$

we obtain

$$\int_{\gamma_1} f\mathrm{d}z = \int_{\gamma_2^-} f\mathrm{d}z$$

as required. □

We now compute one of the fundamental contour integrals. We will show that

$$\int_{\partial B_r(a)} (z-a)^n \mathrm{d}z = \begin{cases} 2\pi\mathrm{i} & n = -1, \\ 0 & n \neq 1, \end{cases} \tag{3.19}$$

where $\partial B_r(a)$ denotes the boundary of the ball of radius r, parametrised counter-clockwise (i.e. positively oriented with respect to $B_r(a)$).

Observe that the result is uniform with respect to r. That is a natural consequence of Theorem 3.31, given than the functions we are integrating only fail to be analytic at one point (at most, depending on n). In fact we could have chosen any curve that wraps around a once and obtain the same result.

Now, to compute the integral above, notice that we can parametrise the curve as $\gamma(t) = a + r\mathrm{e}^{\mathrm{i}t}$, for $t \in [0, 2\pi)$. Therefore we have (since $\gamma'(t) = \mathrm{i}r\mathrm{e}^{\mathrm{i}t}$)

$$\int_{\partial B_r(a)} (z-a)^n \mathrm{d}z = \int_0^{2\pi} (r\mathrm{e}^{\mathrm{i}t})^n \mathrm{i}r\mathrm{e}^{\mathrm{i}t}\mathrm{d}t = \mathrm{i}r^{n+1} \int_0^{2\pi} \mathrm{e}^{\mathrm{i}(n+1)t}\mathrm{d}t.$$

Notice that in the case $n = -1$ that expression equals $2\pi\mathrm{i}$. When $n \neq -1$ notice that we obtain 0, since for all $k \neq 0$ we have

$$\int_0^{2\pi} \mathrm{e}^{\mathrm{i}kt}\mathrm{d}t = \frac{1}{k}\mathrm{e}^{\mathrm{i}kt}\Big|_0^{2\pi} = \frac{1}{k} - \frac{1}{k} = 0.$$

We restate, in the notation that will be most convenient for the next few results, the fundamental integral above in the case $n = -1$, noting that the result does not depend on r. We have

$$\int_{\partial B_r(z)} \frac{1}{w-z}\mathrm{d}w = 2\pi\mathrm{i}.$$

Definition 3.32. *Given a simple closed C^1 curve γ we denote by $I(\gamma)$ the interior region to γ. We denote by $O(\gamma)$ the exterior region to γ.*

Notice that by the deformation of contours Theorem we have

$$\int_\gamma \frac{1}{w-z}\mathrm{d}w = \int_{\partial B_r(z)} \frac{1}{w-z}\mathrm{d}w = 2\pi\mathrm{i} \tag{3.20}$$

for every $z \in I(\gamma)$ and every r sufficiently small so that $B_r(z) \subset I(\gamma)$.

Theorem 3.33. *Let $\gamma : [a,b] \to \mathbb{C}$ be a positively oriented simple closed C^1 curve. Assume that f is analytic in γ and on the interior of γ, $I(\gamma)$. Then*

$$f(z) = \frac{1}{2\pi\mathrm{i}} \int_\gamma \frac{f(w)}{w-z}\mathrm{d}w \qquad \textit{for all } z \in I(\gamma). \tag{3.21}$$

Proof. Fix $z \in I(\gamma)$, and choose r small enough so that $B_r(z) \subset I(\gamma)$. By the deformation of contours theorem we have

$$\frac{1}{2\pi\mathrm{i}} \int_\gamma \frac{f(w)}{w-z}\mathrm{d}w = \frac{1}{2\pi\mathrm{i}} \int_{\partial B_r(z)} \frac{f(w)}{w-z}\mathrm{d}w,$$

reducing the problem to considering γ as a $\partial B_r(z)$. Observe that the integral is the same for every r sufficiently small, and later on we will exploit this fact by talking limits as r tends to zero. For now, we have

$$\frac{1}{2\pi\mathrm{i}} \int_{\partial B_r(z)} \frac{f(w)}{w-z}\mathrm{d}w = \frac{1}{2\pi\mathrm{i}} \int_{\partial B_r(z)} \frac{f(z)}{w-z}\mathrm{d}w + \frac{1}{2\pi\mathrm{i}} \int_{\partial B_r(z)} \frac{f(w)-f(z)}{w-z}\mathrm{d}w =: I + II.$$

Notice that the first integral I equals $f(z)$. Indeed, using (3.20)

$$\frac{1}{2\pi\mathrm{i}} \int_{\partial B_r(z)} \frac{f(z)}{w-z}\mathrm{d}w = f(z)\frac{1}{2\pi\mathrm{i}} \int_{\partial B_r(z)} \frac{1}{w-z}\mathrm{d}w = f(z).$$

All that remains to is to show that $II = 0$. Notice that since f is analytic in $I(\gamma)$, given any $\varepsilon > 0$ we can find r sufficiently small so that

$$|f(w) - f(z)| \leq \varepsilon \qquad \text{for all } w \in \partial B_r(z).$$

We parametrise $\partial B_r(z)$ counterclockwise by $\gamma(t) = z + r\mathrm{e}^{\mathrm{i}t}$ for $t \in [0, 2\pi)$. We have $\gamma'(t) = \mathrm{i}r\mathrm{e}^{\mathrm{i}t}$ and therefore

$$\begin{aligned} |II| = \left| \frac{1}{2\pi\mathrm{i}} \int_{\partial B_r(z)} \frac{f(w)-f(z)}{w-z}\mathrm{d}w \right| &\leq \left| \frac{1}{2\pi\mathrm{i}} \int_0^{2\pi} \frac{f(z+r\mathrm{e}^{\mathrm{i}t}) - f(z)}{r\mathrm{e}^{\mathrm{i}t}} \mathrm{i}r\mathrm{e}^{\mathrm{i}t}\mathrm{d}t \right| \\ &\leq \frac{1}{2\pi} \int_0^{2\pi} |f(z+r\mathrm{e}^{\mathrm{i}t}) - f(z)|\mathrm{d}t \leq \varepsilon. \end{aligned}$$

Since ε is arbitrary we obtain the desired result. ☐

Remark 3.34. *The formula*

$$f(z) = \frac{1}{2\pi\mathrm{i}} \int_\gamma \frac{f(w)}{w - z} \mathrm{d}w \qquad \textit{for all } z \in I(\gamma)$$

has remarkable consequences for analytic functions. First notice that it claims that we can recover the value of f at any point by integration along a curve around that point (provided the curve is sufficiently regular, positively oriented, and contained in $I(\gamma)$). This is a very significant difference with respect to smooth functions in $\mathbb{R}^2$ for example.

Notice that since the curve γ is a compact set, for any point $z \in I(\gamma)$ the expression $w - z$ found in the denominator in Cauchy's formula is bounded away from zero, suggesting that we can differentiate the formula with respect to z to obtain

$$f'(z) = \frac{1}{2\pi\mathrm{i}} \int_\gamma \frac{f(w)}{(w - z)^2} \mathrm{d}w.$$

Of course we need to justify moving the derivative inside the integral sign. We assumed that f was analytic, which means that $f'(z)$ exists. The expression above would produce a formula for it, a way to compute it. The key observation is that without assuming that f has more derivatives it seems that the right hand side can be differentiated arbitrarily many times, which would suggest that f has infinitely many derivatives. This is indeed the case as we will show in the next Theorem.

Theorem 3.35. *Let $\gamma : [a, b] \to \mathbb{C}$ be a positively oriented simple closed C^1 curve. Assume that f is analytic in γ and on the interior of γ, $I(\gamma)$. Then $f^{(n)}(z)$ exists for all $n \in \mathbb{N}$ and the derivative is given by*

$$f^{(n)}(z) = \frac{n!}{2\pi\mathrm{i}} \int_\gamma \frac{f(w)}{(w - z)^{(n+1)}} \mathrm{d}w \qquad \textit{for all } z \in I(\gamma). \tag{3.22}$$

Proof. Notice that Theorem 3.33 would correspond to the case $n = 0$ in the current Theorem. In order to prove the result for $n = 1$ we consider the incremental quotient, and use (3.21) to obtain

$$\frac{f(z+h) - f(z)}{h} = \frac{1}{h} \left[\frac{1}{2\pi\mathrm{i}} \int_\gamma \frac{f(w)}{w - z - h} \mathrm{d}w - \frac{1}{2\pi\mathrm{i}} \int_\gamma \frac{f(w)}{w - z} \mathrm{d}w \right].$$

By the deformation of contours Theorem we can choose γ as $\partial B_{2r}(z)$, with $B_{2r}(z) \subset I(\gamma)$. We have, operating on the right-hand side

$$\frac{f(z+h) - f(z)}{h} = \frac{1}{2\pi\mathrm{i}} \int_{\partial B_{2r}(z)} \frac{f(w)}{(w - z - h)(w - z)} \mathrm{d}w$$

$$= \frac{1}{2\pi\mathrm{i}} \int_{\partial B_{2r}(z)} \frac{f(w)}{(w - z)^2} \mathrm{d}w + \frac{1}{2\pi\mathrm{i}} \int_{\partial B_{2r}(z)} f(w) \left[\frac{1}{(w - z - h)(w - z)} - \frac{1}{(w - z)^2} \right] \mathrm{d}w$$

$$= \frac{1}{2\pi\mathrm{i}} \int_{\partial B_{2r}(z)} \frac{f(w)}{(w - z)^2} \mathrm{d}w + \frac{1}{2\pi\mathrm{i}} \int_{\partial B_{2r}(z)} \left[\frac{h f(w)}{(w - z - h)(w - z)^2} \right] \mathrm{d}w.$$

To conclude the proof all that we need to do is show that the limit of the last integral as h tends to zero is zero, that is (ignoring factors of $2\pi\mathrm{i}$)

$$\lim_{h \to 0} \int_{\partial B_{2r}(z)} \left[\frac{h f(w)}{(w - z - h)(w - z)^2} \right] \mathrm{d}w = 0,$$

and recall that we are able to choose r arbitrarily small without affecting the value of the integrals above.

First we choose $|h| < r$ so that for all $w \in \partial B_{2r}(z)$ we have

$$|w - z - h| \geq |w - z| - |h| > r.$$

Here we have used the reverse triangle inequality in the first case, and the fact that $|w-z| = 2r$ for points $w \in \partial B_{2r}(z)$. Choosing $\gamma(t) = z + 2r\mathrm{e}^{\mathrm{i}t}$ for $t \in [0, 2\pi)$, we have $\gamma'(t) = 2r\mathrm{i}\mathrm{e}^{\mathrm{i}t}$, and therefore $|\gamma'(t)| \leq 2r$. Since f is analytic, in particular it is continuous and therefore there exists $M > 0$ such that $|f(w)| \leq M$ for all $w \in \partial B_{2r}(z)$. Using these facts we have

$$\left| \int_{\partial B_{2r}(z)} \left[\frac{hf(w)}{(w-z-h)(w-z)^2} \right] \mathrm{d}w \right| \leq \int_0^{2\pi} \frac{hM}{r(2r)^2} 2r\mathrm{d}t = \frac{\pi M}{r^2} h,$$

which goes to zero as h goes to zero, proving the result for $n = 1$. The general case is proven by induction. If we assume the result for $n = 1, 2, \cdots, k-1$ we want to prove it for $n = k$. That is, in particular we assume

$$f^{(k-1)}(z) = \frac{(k-1)!}{2\pi\mathrm{i}} \int_\gamma \frac{f(w)}{(w-z)^{(k)}} \mathrm{d}w \qquad \text{for all } z \in I(\gamma).$$

We write the corresponding incremental quotient, just as before

$$\frac{f^{(k-1)}(z+h) - f^{(k-1)}(z)}{h} = \frac{1}{h} \left[\frac{(k-1)!}{2\pi\mathrm{i}} \int_\gamma \frac{f(w)}{(w-z-h)^k} \mathrm{d}w - \frac{(k-1)!}{2\pi\mathrm{i}} \int_\gamma \frac{f(w)}{(w-z)^k} \mathrm{d}w \right].$$

By the deformation of contours Theorem we can choose γ as $\partial B_{2r}(z)$, with $B_{2r}(z) \subset I(\gamma)$. We have, operating on the right-hand side

$$\begin{aligned}
\frac{f^{(k-1)}(z+h) - f^{(k-1)}(z)}{h} &= \frac{(k-1)!}{2\pi\mathrm{i}h} \int_{\partial B_{2r}(z)} \frac{f(w)[(w-z)^k - (w-z-h)^k]}{(w-z-h)^k (w-z)^k} \mathrm{d}w \\
&= \frac{k!}{2\pi\mathrm{i}} \int_{\partial B_{2r}(z)} \frac{f(w)}{(w-z)^{(k+1)}} \mathrm{d}w \\
&+ \frac{(k-1)!}{2\pi\mathrm{i}} \int_{\partial B_{2r}(z)} f(w) \left[\frac{[(w-z)^k - (w-z-h)^k]}{h(w-z-h)^k (w-z)^k} - \frac{k}{(w-z)^{(k+1)}} \right] \mathrm{d}w \\
= \frac{k!}{2\pi\mathrm{i}} \int_{\partial B_{2r}(z)} \frac{f(w)}{(w-z)^{k+1}} \mathrm{d}w &+ \frac{(k-1)!}{2\pi\mathrm{i}} \int_{\partial B_{2r}(z)} f(w) \left[\frac{(w-z)^{k+1} - (w-z-h)^k (w-z) - kh(w-z-h)^k}{h(w-z-h)^k (w-z)^{k+1}} \right] \mathrm{d}w.
\end{aligned} \tag{3.23}$$

As before, all that remains is to show that the last integral tends to zero as h tends to zero. We choose h and the parametrisation as above. The result will follow if we show that

$$\left| \frac{(w-z)^{k+1} - (w-z-h)^k (w-z) - kh(w-z-h)^k}{h} \right| \leq C|h|,$$

where the constant might depend on r. This is the case because, as before $|f| \leq M$ and $|w-z-h| \geq |w-z| - |h| > r$ implies

$$\left| \frac{1}{(w-z-h)^k (w-z)^k} \right| \leq \frac{1}{(2r)^k r^k}.$$

In order to prove (3.23), notice that the binomial formula implies

$$(w-z-h)^k = \sum_{j=0}^{k} \binom{k}{j} (w-z)^{k-j} (-h)^j$$

and therefore

$$\begin{aligned}
&(w-z)^{k+1} - (w-z-h)^k (w-z) - kh(w-z-h)^k \\
&= -\sum_{j=2}^{k} \binom{k}{j} (w-z)^{k+1-j} (-h)^j - kh \sum_{j=1}^{k} \binom{k}{j} (w-z)^{k-j} (-h)^j
\end{aligned}$$

which is of order h^2, proving the result. □

3.3.2 Consequences of Cauchy's Theorem

Theorem 3.36 (Taylor Series Expansion). *Let f be an analytic function on $B_R(a)$ for $a \in \mathbb{C}$, $R > 0$. There there exist unique constants c_n, $n \in \mathbb{N}$ such that*

$$f(z) = \sum_{n=0}^{\infty} c_n (z-a)^n \qquad \text{for all } z \in B_R(a).$$

Moreover, the coefficients c_n are given by

$$c_n = \frac{1}{2\pi \mathrm{i}} \int_\gamma \frac{f(w)}{(w-a)^{n+1}} \mathrm{d}w = \frac{f^{(n)}(a)}{n!},$$

where γ is any positively oriented simple closed curve (piece-wise C^1) that is contained in $B_R(a)$ with $a \in I(\gamma)$.

Proof. Given some $z \in B_R(a)$ we will take γ to be $\partial B_r(a)$ (positively oriented), for r small enough so that $|z-a| < r < R$. We can use the Theorem of deformation of contours to prove the integrals over all curves γ as above are the same. Cauchy's formula (3.21) gives

$$f(z) = \frac{1}{2\pi \mathrm{i}} \int_{\partial B_r(a)} \frac{f(w)}{w-z} \mathrm{d}w. \tag{3.24}$$

Notice that since $|w-a| = r$ and we have chosen r so that $|z-a| < r$ we have $|z-a| < |w-a|$ for all $w \in \partial B_r(a)$. As a result

$$\frac{|z-a|}{|w-a|} < 1$$

and we can use the geometric series expansion to obtain

$$\frac{1}{w-z} = \frac{1}{w-a} \frac{1}{\left(1 - \frac{z-a}{w-a}\right)} = \frac{1}{w-a} \sum_{n=0}^{\infty} \left(\frac{z-a}{w-a}\right)^n.$$

Inserting this expression in (3.24) we obtain

$$f(z) = \frac{1}{2\pi \mathrm{i}} \int_{\partial B_r(a)} f(w) \frac{1}{w-a} \sum_{n=0}^{\infty} \left(\frac{z-a}{w-a}\right)^n \mathrm{d}w.$$

For $w \in \partial B_r(a)$ the series converges absolutely (Weierstrass M-test), and therefore we can exchange the order of the summation and integration to obtain

$$f(z) = \sum_{n=0}^{\infty} \frac{1}{2\pi \mathrm{i}} \int_{\partial B_r(a)} \frac{f(w)}{(w-a)^{n+1}} \mathrm{d}w (z-a)^n = \sum_{n=0}^{\infty} c_n (z-a)^n,$$

obtaining the desired result. It remains to show that the coefficients are unique. Now, assume that $f(z) = \sum_{k=0}^{\infty} b_k (z-a)^k$ for some $b_k \in \mathbb{C}$. We have

$$\begin{aligned}
\int_{\partial B_r(a)} \frac{f(w)}{(w-a)^{n+1}} \mathrm{d}w &= \int_{\partial B_r(a)} \sum_{k=0}^{\infty} b_k (w-a)^k \frac{1}{(w-a)^{n+1}} \mathrm{d}w \\
&= \sum_{k=0}^{\infty} b_k \int_{\partial B_r(a)} (w-a)^{k-n-1} \mathrm{d}w = 2\pi \mathrm{i} b_n,
\end{aligned}$$

where we have used the fundamental integrals, together with the fact that we can commute the summation and integration. This proves that $b_n = c_n$, concluding the proof. ☐

Example 3.37. *We consider an example of a Taylor series. We consider the function $(1+z)^a$ for $a \in \mathbb{C}$ and $|z| < 1$.*

When we consider logarithms we noticed that z^n is well defined for $n \in \mathbb{N}$, but not for any a, without making any specific choice of the argument function. In this case

$$(1+z)^n = \sum_{k=0}^{n} \binom{n}{k} z^k,$$

which is a polynomial of order n, and equals the Taylor series expansion centred at the origin. This series converges for every $z \in \mathbb{C}$, not just $|z| < 1$. However, we defined

$$f(z) = (1+z)^a := \mathrm{e}^{a\mathrm{Log}(1+z)}$$

having made a choice of the argument function defining the logarithm, which meant creating a branch cut where the function was not defined. Choosing the argument in $(-\pi, \pi)$, and since our function is translated (not z^a) we obtained a function that is not defined for $z \in (-\infty, -1]$.

We want to show that in fact a binomial expansion is possible for all $a \in \mathbb{C}$. We know by Taylor's Theorem 3.36 that we have a Taylor expansion. To compute we need to work out the derivatives of $(1+z)^a$. Using the definition we have (for ($a \notin \mathbb{N}$))

$$\left(\mathrm{e}^{a\mathrm{Log}(1+z)}\right)' = \mathrm{e}^{a\mathrm{Log}(1+z)} a(\mathrm{Log}(1+z))' = (1+z)^a \frac{a}{1+z} = a(1+z)^{a-1}.$$

Notice that since by induction we have

$$\frac{\mathrm{d}^k}{\mathrm{d}z^k}\left(\mathrm{e}^{a\mathrm{Log}(1+z)}\right) = a(a-1)\cdots(a-k+1)(1+z)^{a-k}.$$

Therefore we obtain the Taylor series (centred at 0)

$$\sum_{k=0}^{\infty} \frac{a(a-1)\cdots(a-k+1)}{k!} z^k.$$

Notice that the radius of convergence of this series is 1, as we know there are issues for $z \in (-\infty, -1]$. The binomial coefficient, for integer values n and k is

$$\binom{n}{k} = \frac{n!}{(n-k)!k!} = \frac{n(n-1)\cdots(n-k+1)}{k!}$$

and so extending the defition to $n \in \mathbb{C}$ we obtain

$$(1+z)^a = \sum_{k=0}^{\infty} \frac{a(a-1)\cdots(a-k+1)}{k!} z^k = \sum_{k=0}^{\infty} \binom{a}{k} z^k.$$

We can obtain similar expansions centred at different points

$$(1+z)^a = \sum_{k=0}^{\infty} \frac{a(a-1)\cdots(a-k+1)}{k!} (1+z_0)^{a-k} (z-z_0)^k,$$

which would naturally a radius of convergence R equal to the distance from the point z_0 to the half line $\{x \leq -1\}$, where we have made a brach cut for the Log function. You may ignore the issue of the radius of convergence for this series for the exam.

The following result is also a direct consequence of Cauchy's formula.

Theorem 3.38 (Liouville's Theorem)**.** *Let $f : \mathbb{C} \to \mathbb{C}$ be an analytic, bounded function. Then f is constant.*

Proof. Assume that $|f(z)| \leq M$ for all $z \in \mathbb{C}$. Let $a \neq b$ be two points in $\mathbb{C}$. Choose R large enough so that $2\max\{|a|,|b|\} < R$. That means that if we consider $w \in \partial B_R(0)$, that is $|w| = R$ then

$$|w-a| > \frac{R}{2} \qquad\qquad |w-b| > \frac{R}{2}.$$

Since f is analytic in $\mathbb{C}$ we can use Cauchy's formula to compute $f(a)$ and $f(b)$ using $\partial B_R(0)$ as the curve γ (of course positively oriented!). We have

$$f(a) - f(b) = \frac{1}{2\pi \mathrm{i}} \int_{\partial B_R(0)} \frac{f(w)}{w-a} \mathrm{d}w - \frac{1}{2\pi \mathrm{i}} \int_{\partial B_R(0)} \frac{f(w)}{w-b} \mathrm{d}w$$

$$= \frac{1}{2\pi \mathrm{i}} \int_{\partial B_R(0)} f(w) \left(\frac{1}{w-a} - \frac{1}{w-b} \right) \mathrm{d}w = \frac{a-b}{2\pi \mathrm{i}} \int_{\partial B_R(0)} \frac{f(w)}{(w-a)(w-b)} \mathrm{d}w.$$

Therefore

$$|f(a) - f(b)| \leq \frac{|a-b|}{2\pi} \frac{M}{R^2/4} \int_{\partial B_r(0)} 1 \mathrm{d}w = \frac{|a-b| 4M}{R},$$

as $\int_{\partial B_R(0} 1\mathrm{d}w$ is just the length of the curve, which equals $2\pi R$. Notice that since R is arbitrary (provided that it is big enough, as indicated above) we can send R to infinity, showing that $|f(a) - f(b)| = 0$ for any a and b in $\mathbb{C}$, therefore proving that the function is constant. □

A fundamental consequence of Liouville's Theorem is Fundamental Theorem of Algebra.

Theorem 3.39 (Fundamental Theorem of Algebra)**.** *Every non-constant polynomial p on $\mathbb{C}$ has a root, that is, there exists $a \in \mathbb{C}$ such that $p(a) = 0$.*

Proof. We will prove the result by contradiction. Assume that $|p(z)| \neq 0$ for ever $z \in \mathbb{C}$. Define $f : \mathbb{C} \to \mathbb{C}$ by $f(z) = \frac{1}{p(z)}$. Now, since p does not vanish, the function f is analytic in all of $\mathbb{C}$, since it is the composition of two holomorphic functions ($1/z$ is holomophic outside the origin).

Notice that if we assume $p(z) = \sum_{k=0}^{n} c_k z^k$, with $c_n \neq 0$ $(n > 0)$, then at infinity the polynomial behaves like $c_n z^n$, as that is the highest power. That means $|p(z)|$ goes to infinity as z goes to infinity, and satisfies $|p(z)| > 1$ for all $|z| > R$ for some $R > 0$. As a result the function $f(z) = \frac{1}{p(z)}$ is bounded in $\mathbb{C}$. It is less than 1 for all $|z| > R$ based on our analysis of p, and it is bounded on the compact set $|z| \leq R$ since it is continuous.

Liouville's Theorem implies that f is in fact constant, which would force p to be constant, which is a contradiction. □

Theorem 3.40. *Let $f_n : \Omega \to \mathbb{C}$ be a sequence of analytic functions on an open set Ω. If f_n converges uniformly to f, then f is analytic.*

Recall that for a function to be analytic at one point we require that the function be differentiable in a neighbourhood of the point, and therefore the assumption on Ω being open is natural. Being analytic is a local property, and requiring that the uniform convergence holds only on compact sets would suffice.

Proof. Let $z \in \Omega$. Choose $r > 0$ sufficiently small so that $B_r(z) \subset \Omega$. Since f_n is analytic in Ω we can apply Cauchy's formula to obtain

$$f_n(z) = \frac{1}{2\pi \mathrm{i}} \int_{\partial B_r(z)} \frac{f_n(w)}{w-z} \mathrm{d}w.$$

Taking limits as n goes to infinity, and assuming that we can move the limit inside the integral we would obtain

$$f(z) = \frac{1}{2\pi \mathrm{i}} \int_{\partial B_r(z)} \frac{f(w)}{w-z} \mathrm{d}w.$$

We have seen before that this implies that f is differentiable (in fact infinitely differentiable) and obtained an expression for its derivative (see Theorem 3.35). So the only thing left is to justify moving the limit inside the integral. Notice that this is really a one dimensional integral and we can apply the results learnt earlier in the year. Taking $\gamma(t) = z + r\mathrm{e}^{\mathrm{i}t}$ for $t \in [0, 2\pi)$, we have $\gamma'(t) = \mathrm{i}r\mathrm{e}^{\mathrm{i}t}$ and so

$$\int_{\partial B_r(z)} \frac{f_n(w)}{w-z}\mathrm{d}w = \int_0^{2\pi} \frac{f_n(z+r\mathrm{e}^{\mathrm{i}t})}{r\mathrm{e}^{\mathrm{i}t}}\mathrm{i}r\mathrm{e}^{\mathrm{i}t}\mathrm{d}t = \mathrm{i}\int_0^{2\pi} f_n(z+r\mathrm{e}^{\mathrm{i}t})\mathrm{d}t. \tag{3.25}$$

For fixed z, as a function of t we have that $f_n(z+r\mathrm{e}^{\mathrm{i}t})$ converges uniformly to $f(z+r\mathrm{e}^{\mathrm{i}t})$ and applying Theorem 2.16 we can move the limit inside the integral, obtaining

$$\lim_{n\to\infty}\int_{\partial B_r(z)} \frac{f_n(w)}{w-z}\mathrm{d}w = \lim_{n\to\infty}\mathrm{i}\int_0^{2\pi} f_n(z+r\mathrm{e}^{\mathrm{i}t})\mathrm{d}t = \mathrm{i}\int_0^{2\pi} f(z+r\mathrm{e}^{\mathrm{i}t})\mathrm{d}t.$$

Notice that we have (reading the expression (3.25) backwards, now for f instead of for f_n)

$$\mathrm{i}\int_0^{2\pi} f(z+r\mathrm{e}^{\mathrm{i}t})\mathrm{d}t = \int_{\partial B_r(z)} \frac{f(w)}{w-z}\mathrm{d}w,$$

obtaining the result. □

3.3.3 Applications of Cauchy's formula to evaluate integrals in $\mathbb{R}$

We present various examples that illustrate a more general theory (of residues) for computing integrals of functions over $\mathbb{R}$.

Consider for example

$$\int_{-\infty}^{\infty} \frac{1}{1+x^2}\mathrm{d}x.$$

The idea is to consider the contours γ_1 and γ_2 in Figure 3.4.

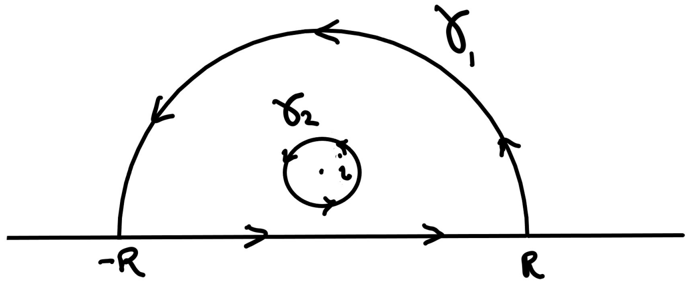

Figure 3.4: Contours

γ_1 is formed by the segment joining $-R$ and R, together with the half circle or radius R. The contour γ_2 is a circle centred at i and of radius $r < 1$. To understand the choice of curves, notice that we can rewrite the integral as

$$\int_{-\infty}^{\infty} \frac{1}{(x-\mathrm{i})(x+\mathrm{i})}\mathrm{d}x.$$

Notice that in the region enclosed by γ_2 the function (the integrand extended to a function on $\mathbb{C}$)

$$f(z) := \frac{1}{(z-\mathrm{i})(z+\mathrm{i})}$$

is analytic except for at $z = \mathrm{i}$. By the deformation of contours Theorem we know that

$$\int_{\gamma_1} f(z)\mathrm{d}z = \int_{\gamma_2} f(z)\mathrm{d}z,$$

since the two curves have the same orientation. Now

$$\int_{\gamma_1} f(z)\mathrm{d}z = \int_{-R}^{R} f(z)\mathrm{d}z + \int_{\text{arc}} f(z)\mathrm{d}z.$$

We parametrise the arc by $R\mathrm{e}^{\mathrm{i}t}$ for $t \in [0, \pi)$. We have

$$\int_{\text{arc}} f(z)\mathrm{d}z = \int_{\text{arc}} \frac{1}{1+z^2}\mathrm{d}z = \int_0^\pi \frac{1}{1+R^2\mathrm{e}^{2\mathrm{i}t}} R\mathrm{i}\mathrm{e}^{\mathrm{i}t}\mathrm{d}t.$$

Therefore

$$\Big|\int_{\text{arc}} f(z)\mathrm{d}z\Big| \le \int_0^\pi \frac{R}{R^2-1}\mathrm{d}t = \pi\frac{R}{R^2-1}.$$

As R tends to infinity the $\int_{\text{arc}} f(z)\mathrm{d}z$ equals zero. Therefore

$$\int_{-\infty}^{\infty} f(z)\mathrm{d}z = \int_{\gamma_1} f(z)\mathrm{d}z = \int_{\gamma_2} f(z)\mathrm{d}z.$$

Now

$$\int_{\gamma_2} f(z)\mathrm{d}z = \int_{\partial B_r(\mathrm{i})} \frac{1}{z+i}\frac{1}{z-i}\mathrm{d}z.$$

Recall that by Cauchy's formula if $g(z)$ is analytic in the interior of a positively oriented curve then

$$\int_\gamma g(z)\frac{1}{z-a}\mathrm{d}z = 2\pi\mathrm{i}g(a).$$

Therefore, taking $g(z) = \frac{1}{z+i}$ we obtain

$$\int_{\partial B_r(\mathrm{i})} \frac{1}{z+i}\frac{1}{z-i}\mathrm{d}z = 2\pi\mathrm{i}\frac{1}{2\mathrm{i}} = \pi,$$

which yields

$$\int_{-\infty}^{\infty} \frac{1}{1+x^2}\mathrm{d}x = \pi.$$

As a second example, let's compute

$$\int_{\infty}^{\infty} \frac{1}{1+x^4}\mathrm{d}x.$$

Notice that the function

$$\frac{1}{1+z^4}$$

has four singularities at the points

$$\mathrm{e}^{\pi\mathrm{i}/4} \qquad \mathrm{e}^{3\pi\mathrm{i}/4} \qquad \mathrm{e}^{-\pi\mathrm{i}/4} \qquad \mathrm{e}^{-3\pi\mathrm{i}/4},$$

and so if we choose a contour similar to the one above (an expanding semi-circle) there will be two singularities in the interior. We obtain the picture 3.5.

Now γ_1 is built out of the line joining $-R$ and R, together with the semi-circle or radius R centred at 0. γ_2 and γ_3 correspond to circles centred at $\mathrm{e}^{3\pi\mathrm{i}/4}$ and $\mathrm{e}^{\pi\mathrm{i}/4}$, oriented clock-wise (positively with respect to both the blue-shaded and yellow-shaded regions). Notice that with those orientations we have

$$\int_{\gamma_1} \frac{1}{1+z^4}\mathrm{d}z + \int_{\gamma_2} \frac{1}{1+z^4}\mathrm{d}z + \int_{\gamma_3} \frac{1}{1+z^4}\mathrm{d}z = 0.$$

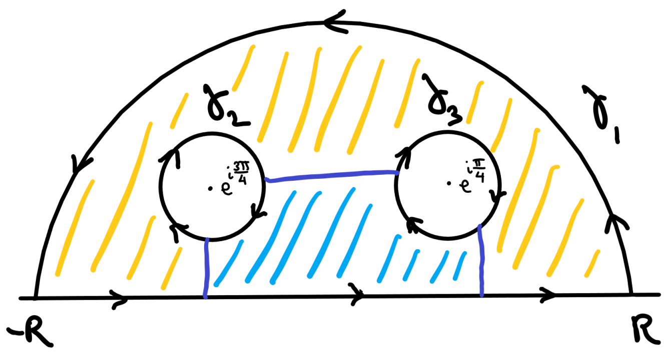

Figure 3.5: Contours

We start by considering the integral over γ_1. Wave

$$\int_{\gamma_1} \frac{1}{1+z^4}\mathrm{d}z = \int_{-R}^{R} \frac{1}{1+z^4}\mathrm{d}z + \int_{\text{arc}} \frac{1}{1+z^4}\mathrm{d}z.$$

We will show that the integral over the arc goes to zero as R goes to infinity. Indeed

$$\left|\int_{\text{arc}} \frac{1}{1+z^4}\mathrm{d}z\right| \leq \int_{\text{arc}} \frac{1}{R^4-1}|\mathrm{d}z| = \frac{\pi R}{R^4-1},$$

which goes to zero as R goes to infinity. Above we have used that $\int_{\text{arc}} |\mathrm{d}z|$ =length(arc)= πR.

We now consider the integral over γ_2. We have (denoting by γ_2^- the anti-clockwise parametrisation of the circle)

$$\int_{\gamma_2} \frac{1}{1+z^4}\mathrm{d}z = -\int_{\gamma_2^-} \frac{1}{(z-\mathrm{e}^{\mathrm{i}\pi/4})(z-\mathrm{e}^{3\mathrm{i}\pi/4})(z-\mathrm{e}^{-\mathrm{i}\pi/4})(z-\mathrm{e}^{-3\mathrm{i}\pi/4})}\mathrm{d}z$$
$$= -\int_{\gamma_2^-} \frac{g(z)}{(z-\mathrm{e}^{3\mathrm{i}\pi/4})}\mathrm{d}z = -2\pi\mathrm{i}g(\mathrm{e}^{3\mathrm{i}\pi/4}),$$

where the above defines g as

$$g(z) = \frac{1}{(z-\mathrm{e}^{\mathrm{i}\pi/4})(z-\mathrm{e}^{-\mathrm{i}\pi/4})(z-\mathrm{e}^{-3\mathrm{i}\pi/4})},$$

and we have used Cauchy's formula as g is analytic inside the curve γ_2^-. Now

$$g(\mathrm{e}^{3\mathrm{i}\pi/4}) = \frac{1}{(\mathrm{e}^{3\mathrm{i}\pi/4}-\mathrm{e}^{\mathrm{i}\pi/4})(\mathrm{e}^{3\mathrm{i}\pi/4}-\mathrm{e}^{-\mathrm{i}\pi/4})(\mathrm{e}^{3\mathrm{i}\pi/4}-\mathrm{e}^{-3\mathrm{i}\pi/4})} = \frac{1}{(-\sqrt{2})(-\sqrt{2}+\sqrt{2}\mathrm{i})(\sqrt{2}\mathrm{i})}.$$

Now we consider the integral over γ_3

$$\int_{\gamma_3} \frac{1}{1+z^4}\mathrm{d}z = -\int_{\gamma_3^-} \frac{1}{(z-\mathrm{e}^{\mathrm{i}\pi/4})(z-\mathrm{e}^{3\mathrm{i}\pi/4})(z-\mathrm{e}^{-\mathrm{i}\pi/4})(z-\mathrm{e}^{-3\mathrm{i}\pi/4})}\mathrm{d}z$$
$$= -\int_{\gamma_3^-} \frac{h(z)}{(z-\mathrm{e}^{\mathrm{i}\pi/4})}\mathrm{d}z = -2\pi\mathrm{i}h(\mathrm{e}^{\mathrm{i}\pi/4}),$$

where the above defines h as

$$h(z) = \frac{1}{(z-\mathrm{e}^{3\mathrm{i}\pi/4})(z-\mathrm{e}^{-\mathrm{i}\pi/4})(z-\mathrm{e}^{-3\mathrm{i}\pi/4})},$$

and we have used Cauchy's formula as h is analytic inside the curve γ_3^-. Now

$$h(\mathrm{e}^{\mathrm{i}\pi/4}) = \frac{1}{(\mathrm{e}^{\mathrm{i}\pi/4} - \mathrm{e}^{3\mathrm{i}\pi/4})(\mathrm{e}^{\mathrm{i}\pi/4} - \mathrm{e}^{-\mathrm{i}\pi/4})(\mathrm{e}^{\mathrm{i}\pi/4} - \mathrm{e}^{-3\mathrm{i}\pi/4})} = \frac{1}{\sqrt{2}(\sqrt{2}\mathrm{i})(\sqrt{2}+\sqrt{2}\mathrm{i})}.$$

Since we have

$$\int_{-\infty}^{\infty} \frac{1}{1+z^4}\mathrm{d}z = -\int_{\gamma_2} \frac{1}{1+z^4}\mathrm{d} \; - \int_{\gamma_3} \frac{1}{1+z^4}\mathrm{d}z$$

we obtain

$$\int_{-\infty}^{\infty} \frac{1}{1+z^4}\mathrm{d}z = 2\pi\mathrm{i}\frac{1}{(-\sqrt{2})(-\sqrt{2}+\sqrt{2}\mathrm{i})(\sqrt{2}\mathrm{i})} + 2\pi\mathrm{i}\frac{1}{\sqrt{2}(\sqrt{2}\mathrm{i})(\sqrt{2}+\sqrt{2}\mathrm{i})} = \frac{\pi\sqrt{2}}{2}.$$

In addition to being able to integrate quotients involving polynomials, we can integration some trigonometric functions. For example

$$\int_{-\infty}^{\infty} \frac{\cos(3x)}{4+x^2}\mathrm{d}x.$$

We can rewrite this integral as

$$\mathbf{Re}\int_{-\infty}^{\infty} \frac{\mathrm{e}^{3\mathrm{i}z}}{(z-2\mathrm{i})(z+2\mathrm{i})}\mathrm{d}z,$$

and we can actually drop the $\mathbf{Re}$ part as the imaginary part will be an odd integrand and it will vanish. We consider the contours (notice they are both oriented counter-clockwise)

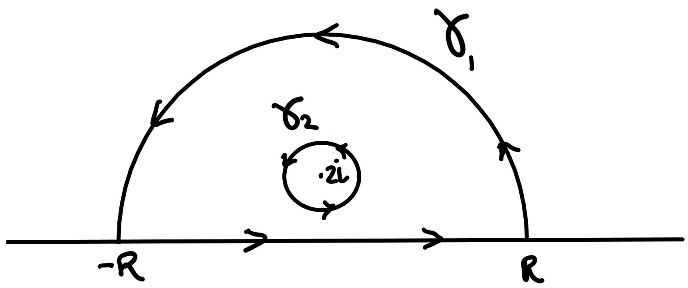

Figure 3.6: Contours

As before,

$$\int_{\gamma_1} \frac{\mathrm{e}^{3\mathrm{i}z}}{(z-2\mathrm{i})(z+2\mathrm{i})}\mathrm{d}z = \int_{\gamma_2} \frac{\mathrm{e}^{3\mathrm{i}z}}{(z-2\mathrm{i})(z+2\mathrm{i})}\mathrm{d}z\mathrm{d}z.$$

Also

$$\int_{\gamma_1} \frac{\mathrm{e}^{3\mathrm{i}z}}{(z-2\mathrm{i})(z+2\mathrm{i})}\mathrm{d}z = \int_{-R}^{R} \frac{\mathrm{e}^{3\mathrm{i}z}}{(z-2\mathrm{i})(z+2\mathrm{i})}\mathrm{d}z + \int_{\text{arc}} \frac{\mathrm{e}^{3\mathrm{i}z}}{(z-2\mathrm{i})(z+2\mathrm{i})}|\mathrm{d}z|.$$

We consider first the integral over the arc (half circle of radius R). We have, for $R >> 4$

$$\left|\int_{\text{arc}} \frac{\mathrm{e}^{3\mathrm{i}z}}{(z-2\mathrm{i})(z+2\mathrm{i})}\mathrm{d}z\right| \le \int_{\text{arc}} \frac{|\mathrm{e}^{3\mathrm{i}z}|}{|z^2+4|}|\mathrm{d}z| \le \int_{\text{arc}} \frac{\mathrm{e}^{-3\,\mathbf{Im}\,z}}{R^2-4}|\mathrm{d}z| \le \frac{\pi R}{R^2-4} \xrightarrow[R\to\infty]{} 0,$$

where we have used that along the arc, $\mathbf{Im}\,z \ge 0$ and so $\mathrm{e}^{-3\,\mathbf{Im}\,z} \le 1$. Now for γ_2 (remember it is oriented anti-clockwise)

$$\int_{\gamma_2} \frac{\mathrm{e}^{3\mathrm{i}z}}{(z-2\mathrm{i})(z+2\mathrm{i})}\mathrm{d}z = \int_{\gamma_2} \frac{g(z)}{z-2\mathrm{i}}\mathrm{d}z = 2\pi\mathrm{i}g(2\mathrm{i}),$$

where

$$g(z) = \frac{e^{3iz}}{z+2i}dz$$

and we have used Cauchy's formula since g is analytic inside γ_2. We have

$$g(2i) = \frac{e^{-6}}{4i}.$$

Therefore

$$\int_{-\infty}^{\infty} \frac{e^{3iz}}{(z-2i)(z+2i)}dz = \int_{\gamma_1} \frac{e^{3iz}}{(z-2i)(z+2i)} = \int_{\gamma_2} \frac{e^{3iz}}{(z-2i)(z+2i)} = 2\pi i g(2i) = \frac{\pi}{2}e^{-6}.$$

We use a similar approach for

$$\int_{-\infty}^{\infty} \frac{x \sin x}{1+x^2}dx = \frac{1}{i}\int_{-\infty}^{\infty} \frac{ze^{iz}}{1+z^2}dz.$$

Notice the real part of the integral vanishes as the integrand is odd (hence dividing the i). We consider the following contours of integration We consider the contours (notice they are both oriented counter-clockwise) By Cauchy's Theorem since the integrand is analytic in the region between the curves we

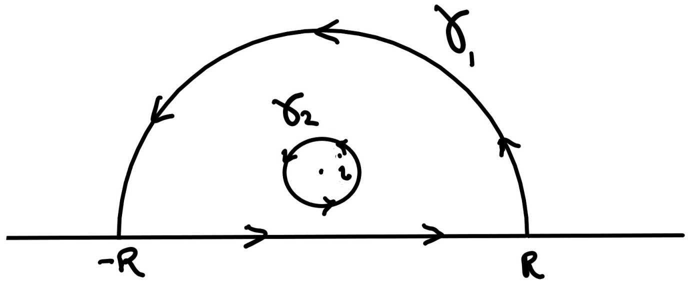

Figure 3.7: Contours

have

$$\int_{\gamma_1} \frac{ze^{iz}}{1+z^2}dz = \int_{\gamma_2} \frac{ze^{iz}}{1+z^2}dz.$$

Now for γ_2

$$\int_{\gamma_2} \frac{ze^{iz}}{(z-i)(z+i)}dz = \int_{\gamma_2} \frac{h(z)}{(z-i)}dz = 2\pi i h(i),$$

for

$$h(z) = \frac{ze^{iz}}{z+i},$$

and so

$$\int_{\gamma_2} \frac{ze^{iz}}{(z-i)(z+i)}dz = 2\pi i \frac{ie^{-1}}{2i} = \frac{\pi}{e}i.$$

We now consider the integral over γ_1. We look at the integral along the arc.

$$\left|\int_{\text{arc}} \frac{ze^{iz}}{1+z^2}dz\right| = \left|\int_0^{\pi} \frac{Re^{it}e^{-R\sin t + Ri\cos t}}{1+R^2e^{2it}} Rie^{it}dt\right| \leq \int_0^{\pi} \frac{R^2}{R^2-1}e^{-R\sin t}dt$$

$$\leq 2\int_0^{\pi} \mathrm{e}^{-R\sin t}\mathrm{d}t = 4\int_0^{\pi/2} \mathrm{e}^{-R\sin t}\mathrm{d}t \leq 4\int_0^{\pi/2} \mathrm{e}^{-R2t/\pi}\mathrm{d}t = -4\frac{\pi}{2R}\mathrm{e}^{-R2t/\pi}\Big|_0^{\pi/2}$$
$$= \frac{-2\pi}{R}\left[\mathrm{e}^{-R} - 1\right] \xrightarrow[R\to\infty]{} 0.$$

Therefore

$$\int_{-\infty}^{\infty} \frac{x\sin x}{1+x^2}\mathrm{d}x = \frac{1}{\mathrm{i}}\int_{\gamma_1} \frac{z\mathrm{e}^{\mathrm{i}z}}{(z-\mathrm{i})(z+\mathrm{i})}\mathrm{d}z = \frac{1}{\mathrm{i}}\int_{\gamma_2} \frac{z\mathrm{e}^{\mathrm{i}z}}{(z-\mathrm{i})(z+\mathrm{i})}\mathrm{d}z = \frac{\pi}{\mathrm{e}}.$$

As the final example we consider an integrand that has a singularity along the natural path of integration

$$\int_{-\infty}^{\infty} \frac{\sin x}{x}\mathrm{d}x.$$

If we complexify the integrand, we are left with

$$\frac{1}{\mathrm{i}}\int_{-\infty}^{\infty} \frac{\mathrm{e}^{\mathrm{i}z}}{z}\mathrm{d}z,$$

since the real part of the integrand is odd. Since the denominator vanishes for a point in the real axis we need to modify the contours we consider.

The contour is formed of 4 curves, and since the integrand is analytic we know that

$$\int_{\gamma_1} \frac{\mathrm{e}^{\mathrm{i}z}}{z}\mathrm{d}z + \int_{\gamma_2} \frac{\mathrm{e}^{\mathrm{i}z}}{z}\mathrm{d}z + \int_{\gamma_3} \frac{\mathrm{e}^{\mathrm{i}z}}{z}\mathrm{d}z + \int_{\gamma_4} \frac{\mathrm{e}^{\mathrm{i}z}}{z}\mathrm{d}z = 0.$$

Now for γ_1

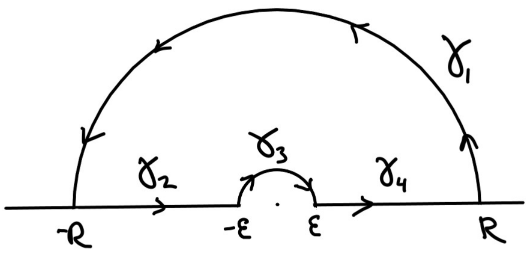

Figure 3.8: Contours

$$\int_{\gamma_1} \frac{\mathrm{e}^{\mathrm{i}z}}{z}\mathrm{d}z = \int_0^{\pi} \frac{\mathrm{e}^{\mathrm{i}R\cos t - R\sin t}}{R\mathrm{e}^{\mathrm{i}t}}\mathrm{i}R\mathrm{e}^{\mathrm{i}t}\mathrm{d}t$$

and so

$$\left|\int_{\gamma_1} \frac{\mathrm{e}^{\mathrm{i}z}}{z}\mathrm{d}z\right| \leq \int_0^{\pi} \mathrm{e}^{R\sin t}\mathrm{d}t \xrightarrow[R\to\infty]{} 0$$

as we have seen before. As for γ_3, since it is oriented clock-wise

$$\int_{\gamma_3} \frac{\mathrm{e}^{\mathrm{i}z}}{z}\mathrm{d}z = -\int_{\gamma_3^-} \frac{\mathrm{e}^{\mathrm{i}z}}{z}\mathrm{d}z = -\int_0^{\pi} \frac{\mathrm{e}^{\mathrm{i}\varepsilon e^{\mathrm{i}t}}}{\varepsilon\mathrm{e}^{\mathrm{i}t}}\mathrm{i}\varepsilon\mathrm{e}^{\mathrm{i}t}\mathrm{d}t = -\mathrm{i}\int_0^{\pi} \mathrm{e}^{\mathrm{i}\varepsilon\cos t}\mathrm{e}^{-\varepsilon\sin t}\mathrm{d}t \xrightarrow[\varepsilon\to 0]{} -\pi\mathrm{i}.$$

Therefore

$$\int_{-\infty}^{\infty} \frac{\sin x}{x}\mathrm{d}x = \frac{1}{\mathrm{i}}\int_{-\infty}^{\infty} \frac{\mathrm{e}^{\mathrm{i}z}}{z}\mathrm{d}z$$
$$= \lim_{\varepsilon\to 0}\lim_{R\to\infty} \frac{1}{\mathrm{i}}\left[\int_{\gamma_2} \frac{\mathrm{e}^{\mathrm{i}z}}{z}\mathrm{d}z + \int_{\gamma_4} \frac{\mathrm{e}^{\mathrm{i}z}}{z}\mathrm{d}z\right] = \lim_{\varepsilon\to 0}\lim_{R\to\infty}\left[-\frac{1}{\mathrm{i}}\int_{\gamma_1} \frac{\mathrm{e}^{\mathrm{i}z}}{z}\mathrm{d}z - \int_{\gamma_3} \frac{1}{\mathrm{i}}\frac{\mathrm{e}^{\mathrm{i}z}}{z}\mathrm{d}z\right] = \pi.$$

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